Available online at www.sciencedirect.com

SCIENCE DIRECT®

Economics Letters 90 (2006) 183–188

economics letters

www.elsevier.com/locate/econbase

Testing to confirm that Leontief–Sraffa matrix equations for input/output must obey constancy of returns to scale

Paul A. Samuelson *, Erkko M. Etula

Massachusetts Institute of Technology, Department of Economics, E52-383, 50 Memorial Drive, Cambridge, MA 02142, United States

Received 6 October 2004; received in revised form 6 June 2005; accepted 13 July 2005
Available online 28 December 2005

Abstract

Multiple proofs that competitive intertemporal equilibrium and Leontief–Sraffa input/output matrix relations do presuppose constant returns to scale.

Keywords: Surplus technologies; Sraffian standard commodity

JEL classification: B1; B1; B5; B10; B20; C6

Sraffa (1960, Preface, pp. v, vi) announces in the beginning that his readers are free to assume that returns to scale are not constant. Since, however, the matrix algebra of both Leontief (1941) and Sraffa (1960) (and Ramsey too) do adhere strictly to the first-degree homogeneous axiom, we believe this has been a case where Homer nodded. (Keynes had warned Sraffa in 1928 that if he did not insist on constancy of scale returns, he should warn his readers of that. We suspect that Frank Ramsey, who had sketched for Sraffa the matrix equations he would need, may have prompted Keynes to make this intervention.)

The matter can be put to unambiguous testings. We do that here several times and in every case deduce that without constancy of scale returns the 1960 apparatus becomes both logically incoherent and empirically irrelevant for that competitive distribution equilibrium long sought by Ricardo (1817).

* Corresponding author. Tel.: +1 617 253 3368; fax: +1 617 253 0560.

0165-1765/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.econlet.2005.07.024

1. Testing 1: Sraffa's (1960, p. 3, n. 1) first cryptic footnote

Tableaux **I** and **I′** each involve wheat and iron sectors that produce but one product out of earlier period wheat and iron inputs. **I** is by definition a "bare subsistence technology," just capable of self-replacement (and therefore incapable of any positive growth or of any positive net wheat or iron consumptions); in addition, **I** could not afford to pay any positive real wage to any labor input it might need, nor can it afford to pay any positive safe interest rate.

I′ by contrast fails to be able to replace its iron inputs. Sraffa's point is that, nevertheless, such an **I′** might still be able to become a bare subsistence economy by judicious balanced (!) increases in the scale of the iron sector, while the wheat sector is at the same time being reduced in scale by balanced decreases in its output and its input(s).

For clarity, we replace the two-digit 1960 technical coefficiencies by small one-digit low numbers. Our self-explaining notations write out the bare subsistence Tableau **I** as follows:

Tableau **I**

$$\begin{aligned}&\text{wheat sector: } \underline{2} \text{ of } \underline{K}_1(t)^{\text{wh}} \;\&\; \underline{2} \text{ of } \underline{K}_2(t)^{\text{wh}} \rightarrow \text{gross } \overline{Q}_1(t+1) = \overline{2} \text{ of wheat}\\&\text{iron sector: } \underline{0} \text{ of } \underline{K}_1(t)^{\text{ir}} \;\&\; \underline{2} \text{ of } \underline{K}_2(t)^{\text{ir}} \rightarrow \text{gross } \overline{Q}_2(t+1) = \overline{4} \text{ of iron.}\end{aligned} \tag{1.1}$$

This does satisfy the definition of a "bare subsistence" technology because it is just capable of replacing both its needed wheat inputs and its iron inputs:

$$\underline{2} + \underline{0} = \overline{2}, \; \underline{2} + \underline{2} = \overline{4}. \text{ QED} \tag{1.2}$$

Note that **I** cannot produce any positive stationary consumptions of either good:

$$\overline{C}_1(t+1) \equiv \overline{0}, \; \overline{C}_2(t+1) \equiv \overline{0}. \tag{1.3}$$

Note too that Tableau **I** is incapable of any positive growth. And if any direct labor had been needed for **I**, their real wage rates would have to be zero. ("Living on thin air.") Readers will also perceive that there could be no positive safe interest, i^*, in **I**.

Sraffa (1960, p. 3, n. 1) cryptically makes a correct point. There could be an **I′** technology that is not itself in the bare subsistence mode but which might be put into it by feasible balanced scale (!) changes in wheat and iron productions. Tableau **I′** is an alternative technology that can be tested in this regard:

Tableau **I′**

$$\begin{aligned}&\text{wheat: } \underline{3} \text{ of } \underline{K}_1(t)^{\text{wh}} \;\&\; \underline{3} \text{ of } \underline{K}_2(t)^{\text{wh}} \rightarrow \overline{Q}_1(t+1)' = \overline{3}\\&\text{iron: } \underline{0} \text{ of } \underline{K}_1(t)^{\text{ir}} \;\&\; \underline{1} \text{ of } \underline{K}_2(t)^{\text{ir}} \rightarrow \overline{Q}_2(t+1)' = \overline{2}.\end{aligned} \tag{2.1}$$

I′ is seen to be incapable of reproducing the iron inputs it uses up. And so its society is doomed to decay exponentially through time and go extinct.

$$\overline{2} \text{ of iron} < (\underline{3} + \underline{1}) \text{ of iron. QED} \tag{2.2}$$

Not to worry, says Piero. Double in balanced scale, all of (**I′**)'s technical iron-sector coefficients while reducing all of its wheat-sector coefficients by a balanced one-third. Then abracadabra, presto, **I′** is transformed into being an exact clone of **I**. QED. Reader, verify this.

What is the moral of the story? It tells that an author who believes himself to be free of the Axiom of Constant Returns to Scale is unconsciously in its spell everywhere in the 1960 book.[1]

Readers can work out why increasing returns to scale would–à la Smith (1776), Young (1928), Graham (1923), Ohlin (1933) and Matthews (1940)–lead to monopolistic imperfections of competition with no uniformity of profit rates or of wage rates in the divergent sectors of wheat, iron, etc. Sraffa (1926) had already made that point about imperfect competition. Astute readers of the 1960 classic would be impelled to stop reading the book after its page 3 if they accepted its author's delusion that his linear matrix algebra equations obeyed anything other than constant returns to scale.

The same goes for decreasing returns to scale. As readers of Wicksell's Lectures know, that would lead to an infinity of infinitesimal firms just to produce two units of wheat and four units of iron! (Besides, in a convex technology, one could replace decreasing scale returns by constant returns simply by handling a doubling of scale by doing twice side by side the original status quo.) Because Sraffa was so allergic to Clark–Wicksteed marginalisms, which did posit first-degree-homogeneous production functions, he not only gave himself liberty to assume second degree or one-half degree or locally changing degrees of homogeneity. Yet, as the following further testings will each demonstrate, the Leontief–Sraffa matrix relations do always postulate constant scale returns.

2. Testing 2: how a surplus economy maintains the same real prices and interest rates independently of changing consumer demand tastes

From page 3 on Sraffa (1960) goes beyond Subsistence Economies to Surplus Economies capable of producing stationary-state positive consumptions of wheat and iron. Such cases are posited to require Labor inputs to work along with wheat and iron produced inputs. Ignoring joint-products complications, our Tableaux **II**, **II′**, **II″**, etc. will, like the earlier Tableaux **I** and **I′** provide testings that rebut the cogency of departing from the Axiom of Constant Returns to Scale.

For brevity, to adduce the surplus Tableau **II**, we retain Tableau **I**'s input coefficients but double its output coefficients. Also, like Sraffa, we stipulate that at least one sector will require some positive exogenous Labor. Our numeraire convention is that total exogenous labor will be anchored at unity, so that

$$\begin{aligned}&\underline{L}(t)\equiv\underline{L}(t)^{\mathrm{wh}}+\underline{L}(t)^{\mathrm{ir}}\equiv 1\equiv\overline{L}(t+1)\\&\left[\underline{K}_1(t),\underline{K}_2(t)\right]\equiv\left[\overline{K}_1(t+1),\overline{K}_2(t+1)\right]\equiv\left[K_1^{\mathrm{e}},K_2^{\mathrm{e}}\right]\\&\left[\overline{C}_1(t+1),\overline{C}_2(t+1)\right]=\left[C_1^*,C_2^*\right].\end{aligned}\tag{3}$$

[1] Historical digression. Returning from the stellar 1962 Paris Econometrica Conference on Risk, Samuelson spent a few days visiting Kings College, Cambridge. Walking near the Cam with Sraffa, he adverted to the above page 3 footnote problem. It was suggested to Sraffa that he should apply the Hawkins-Simons (1949) three determinant tests to separate out **I′** candidates from other technologies incapable of being transferred into the **I**-like mode. (The H-S tests do presuppose strict constant returns to scale.) But always Piero preferred his own trial-and-error methods. It is of interest that not until the 1958 Korfu meeting of the International Economic Association did Piero mention that for a third of a century he had been working on a soon-to-be published book on capital theory. One further remark. Anyone who got a glimpse of an early sketch for Sraffa by Ramsey will have found that its matrix equations all did postulate constant returns to scale!

In Eq. (3) the K_j^e can be taken as exogenous endowments for brevity.

Tableau **II**

$$\begin{aligned}&\underline{.5} \text{ of } \underline{L}(t)^{wh} \,\&\, \underline{2} \text{ of } \underline{K}_1(t)^{wh} \,\&\, \underline{2} \text{ of } \underline{K}_2(t)^{wh} \rightarrow \overline{Q}_1(t+1) = \overline{4} = [\underline{2} + \underline{0}] + [\overline{2} \text{ of } \overline{C}_1(t+1)]\\ &\underline{.5} \text{ of } \underline{L}(t)^{ir} \,\&\, \underline{0} \text{ of } \underline{K}_1(t)^{ir} \,\&\, \underline{2} \text{ of } \underline{K}_2(t)^{ir} \rightarrow \overline{Q}_2(t+1) = \overline{8} = [\underline{2} + \underline{2}] + [\overline{4} \text{ of } \overline{C}_2(t+1)].\end{aligned} \quad (4)$$

To explore Sraffa's distinction between a Basic Product (like iron in this example) and a Non-Basic (like wheat in this example), we singularly stipulated that iron had zero (!) need for wheat input. Because both goods did need iron inputs, by definition iron is a Sraffian Basic. But because not every good needs some wheat input, wheat is a Non-Basic.

Whenever rentier owners of permanent $[K_1^e, K_2^e]$ are to receive zero interest return as their share of the Surplus net harvest(s), i^* will equal zero. In such a Schumpeter (1912) Golden Rule state of maximal real wages, competitive equating of supply and demand for L^{wh} and L^{ir} will be able to achieve full employment at unique arbitrage-proof real wage rates for wheat and for iron.

$$(W/P_1)^{go} = \overline{2}/\underline{.5} = 4 \text{ of wheat} \quad (5.1)$$

$$(W/P_2)^{go} = \overline{4}/\underline{.5} = 8 \text{ or iron} \quad (5.2)$$

$$\therefore\ (P_2/P_1)^{go} = \left(W/P_1^{go}\right)/(W/P_2)^{go} = 4/8 = 1/2. \quad (5.3)$$

At the other pole, where i^* is maximal rather than zero, both wage rates must of course be zero. Since we had cunningly doubled both $\bar{Q}_1$ and $\bar{Q}_2$ in **II** over (**I**)'s gross outputs, it will be evident that **II** can sustain a maximal balanced growth rate of 100%. This sets Max i^* at

$$i^* = 1.00. \quad (5.4)$$

In between $i^*=0$ and $i^*=1.00$–as at 10%, 50%, or 80%–Sraffa's (1960, Ch. II) linear equations will (when solved) provide the two Ricardo–Hollander inverse tradeoff formulas for both iron and wheat real wage rates versus the i^* interest rate.

What if all the consumers in this society cared only for wheat as the final consumption desideratum? Iron they then want only as it is needed to help labor and wheat inputs produce consumed wheat. We can now test how Sraffa's Chapter I, footnote procedure will transform **II** into the following **II**′

Tableau **II**′

$$\begin{aligned}&\underline{.75} \text{ of } \underline{L}(t)^{wh} \,\&\, \underline{3} \text{ of } \underline{K}_1(t)\underline{L}^{wh} \,\&\, \underline{3} \text{ of } \underline{K}_2(t)^{wh} \rightarrow \overline{Q}_1(t+1)' = [\underline{3} + \underline{0}] + [\overline{3} \text{ of } \overline{C}_1(t+1)'].\\ &\underline{.25} \text{ of } \underline{L}(t)^{ir} \,\&\, \underline{0} \text{ of } \underline{K}_1(t)L^{ir} \,\&\, \underline{1} \text{ of } \underline{K}_2(t)L^{ir} \rightarrow \overline{Q}_2(t+1)' = [\underline{3} + \underline{1}] + [\overline{0} \text{ of } \overline{C}_2(t+1)]'\end{aligned} \quad (6)$$

It will be evident that (**II**′) scale changes are balanced changes from **II**'s: iron's scale is halved from **II**'s scale; wheat's scale is increased by 50% over **II**'s scale. Scale returns are strictly constant in each sector! Had they not been, a new Tableau **II**″ would be distinctly different from Tableau **II**′. And from it, the linear equations prescribed by Sraffa (1960, Ch. ii) would not (!) generate the same Sraffian real prices that Tableau **II** had done. Non-constancy of scale returns would mandate that mere changes in demand tastes could lead to an infinity of different Ricardo–Hollander–Sraffa inverse tradeoffs between real wage and interest rates—a reductio ad absurdum.

3. Testing 3: how non-constancy of scale returns leads to an infinity of different Sraffa Standard Commodities rather than to his one

Sraffa's odd brain child of the "Standard Commodity", which he developed in the hope of explicating Ricardo's unexpressed "meaning", will when stripped of its formal definition as a one-sided eigenvector of the input/output matrix, become by definition one that ignores all non-basics. In our Machiavellian examples of **I**, **I′**, **II**, **II′** the Sraffa's Standard vector boils down to changing from **II**'s ($\underline{L}^{wh}$, $\underline{L}^{ir}=1-\underline{L}^{wh}$)=(.5,.5) to **II″**'s ($\underline{L}^{wh}$, $\underline{L}^{ir}$)= (0,1).

Tableau **II″**

$$\begin{aligned}&\text{wheat: } \underline{0} \text{ of } \underline{L}(t)^{wh} \;\&\; \underline{0} \text{ of } \underline{K}_1(t)^{wh} \;\&\; \underline{0} \text{ of } \underline{K}_2(t)^{wh} \rightarrow \overline{0} \text{ of } \overline{Q}_1(t+1)'' = \overline{0} \text{ of } \overline{C}_1(t+1)''\\ &\text{iron: } \underline{1} \text{ of } \underline{L}(t)^{ir} \;\&\; \underline{0} \text{ of } \underline{K}_1(t)^{ir} \;\&\; \underline{4} \text{ of } \underline{K}_2(t)^{ir} \rightarrow \overline{8} \text{ of } \overline{Q}_2(t+1)'' = [\underline{0}+\underline{4}] + [\overline{4} \text{ of } \overline{C}_2(t+1)''].\end{aligned} \tag{7}$$

Notice the contrast in scale between

$$(\mathbf{II})\text{'s} \quad \underline{.5} \text{ of } \underline{L}(t)^{ir} \;\&\; \underline{0} \text{ of } \underline{K}_1(t)^{ir} \;\&\; \underline{2} \text{ of } \underline{K}_2(t)^{ir} \rightarrow \overline{Q}_2(t+1) = \overline{4} \tag{8.1}$$

$$(\mathbf{II}'')\text{'s} \quad \underline{1} \text{ of } \underline{L}(t)^{ir} \;\&\; \underline{0} \text{ of } \underline{K}_1(t)^{ir} \;\&\; \underline{4} \text{ of } \underline{K}_2(t)^{ir} \rightarrow \overline{Q}_2(t+1)'' = \overline{8}. \tag{8.2}$$

Any reader who took seriously Sraffa's invitation to assume non-constant returns to scale could not legitimately deduce (8.2) from (8.1) and would be seduced into accepting an infinity of different Standard commodities.

It may be noted that post-Sraffians, with a few honorable exceptions, fail to recognize that the Sraffa brain child contributes nought to understanding any complete Ricardian distribution model; and, worse than that, it tempts to misunderstandings.

4. Concluding remarks

When an autodidact nods, perhaps no explanation is needed. The nod explains itself. In the present case, an important clue might come from the possible fact that over several decades Sraffa seems often to have confused "constant scale returns" with "Marshallian constant costs." This could explain his 1960 Preface, p. vi, remark:

> The temptation to presuppose constant returns ... was experienced by the author himself . . . and it led him [Sraffa] in 1925 to argue that only the case of constant returns was generally consistent with the premises of economic theory.

What we and mainstream theorists understood the young Sraffa (1925, 1926) to be arguing was that horizontal Marshallian *ss* supply curves were somehow better than rising *ss* curves. Did the author remember, in 1960, that those rising (1926) *ss* curves did not at all imply non-constant returns to scale? (When Smith–Ricardo contemplated exchange between Good A produced by direct labor alone and Good B produced by labor and homogeneous land, they expected elevated consumers tastes for B as against A would raise the equilibrium $(P_B/P_A)^*$ even though both A and B did enjoy constant returns to scale. In 1960, wherever Sraffa wrote "constant returns," Samuelson penned in his book copy "constant returns [to scale]." Knowing that Sraffa incurred some memory problems in his later decades, Samuelson hesitated to quiz him on this detail. Instead he wrote to Maurice Dobb, asking him to sound out Piero on the question, "When you write 'constant returns' do you intend that always to be 'constant returns to scale?'" Dobb reported back Piero as saying, "What else could I mean?" This suggests that his confusion on this point started early and persisted long. It might also clarify how Sraffa could come to write that classical economics differed from mainstream 20th century by its denying the necessity of scale returns constancy.)

References

Graham, F., 1923. The theory of international values re-examined. Quarterly Review of Economics 38, 54–86.

Leontief, W., 1941. The Structure of American Economy, 1919–1929. Harvard University Press, Cambridge, MA.

Matthews, R., 1940. Reciprocal demand and increasing returns. Review of Economic Studies 17, 149–158.

Ohlin, B., 1933. Interregional and International Trade. Harvard University Press, Cambridge, Mass.

Ricardo, D., 1817. The works and correspondence of David Ricardo, P. Sraffa, ed., with M.H. Dobb, On the Principles of Political Economy and Taxation, vol. 1. Cambridge University Press, Cambridge. 1951.

Schumpeter, J. A., 1912. *Theorie der wirtschaftlichen Entwicklung*, Leipzig: Duncker & Humblot. Translated as *The Theory of Economic Development*, Cambridge, Mass.: Harvard University Press, 1934.

Smith, A., 1776. *An Inquiry into the Nature and Causes of the Wealth of Nations*. Ed. E. Cannan, New York: The Modern Library, 1937.

Sraffa, P., 1925. Sulle relazioni tra costi e quantità prodotta. Annali di Economica II, 277–328.

Sraffa, P., 1926. The laws of returns under competitive conditions. Economic Journal 36, 535–550.

Sraffa, P., 1960. Production of Commodities by Means of Commodities, Prelude to a Critique of Economic Theory. Cambridge University Press, Cambridge.

Young, A., 1928. Increasing returns and economic progress. Economic Journal 38, 527–542.