Prices and Index Numbers
Author(s): R. S. Padan
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PRICES AND INDEX NUMBERS.

THERE is today considerable confusion as to the significance or value of tables of prices and index numbers. The averages figuring in these tables have been a fruitful source of controversy. The arithmetic mean is most commonly employed, but W. S. Jevons advocated and employed the geometric method,[1] and F. Y. Edgeworth has suggested the median[2] as preferable to any other means for representing the average, at least, in a certain type of cases. The majority of statisticians favor the use of index numbers for determining the movement of prices, but M. G. Mulhall[3] regards the results given by index numbers as utterly fallacious, and N. G. Pierson,[4] by applying a criterion that appears to show glaring inconsistencies, has found justification for discrediting all attempts at discovering movements of prices.

Under the circumstances it will perhaps not be out of place to re-examine the premises to these conflicting views. With this in view this paper presents the following theses:

1. The arithmetic is the only rigorous method for computing averages.

2. Present systems of index numbers are defective, but the remedy is simple.

3. Periodic movement of prices can be accurately presented in the case of single commodities; in the case of a greater number of commodities the movement can be shown only when the quantities in the various periods are proportional.

I. METHODS OF TAKING AVERAGES.

In preparing tables of prices and index numbers it has been necessary to reduce lists of figures representing prices for various months or years, or for various commodities, to average values.

[1] *Investigations in Currency and Finance*, pp. 23, 24, 120 *et seq.*

[2] *Reports of the British Association for the Advancement of Science.*

[3] *History of Prices* (London, 1885), p. 7.

[4] *Economic Journal*, March 1896, p. 131.

Primarily it is a matter of importance that averages have been accurately computed. It would seem that for so simple a matter as taking an average there would be little excuse for uncertainty or confusion. To be sure, where considerations of weighting the figures enter, the question becomes somewhat complex. But aside from any such complexity, there has been by no means unanimity or certainty of opinion as to the proper method of taking an average. Jevons persisted in using what is known as the geometric method, and the influence of his name has been sufficient to perpetuate a considerable measure of doubt as to the merits of the various claims. To make matters worse, Jevons introduced into the controversy a new potential candidate for the honor of representing the average, namely, the harmonic mean. We have thus to consider three kinds of mean in relation to average—arithmetic, geometric, and harmonic. An arithmetic series, as is well known, is one that has a constant difference between the successive members, as 5, 8, 11, 14, etc., with the constant difference 3. A geometric series has a constant ratio between successive members, as 2, 4, 8, 16, etc., with the constant ratio 2. The harmonic series is not so simply stated as either of the above. For this discussion, the relation of this series to the arithmetic will afford the most intelligible and suggestive definition. If we express an arithmetic series in the form of fractions ($\frac{3}{1}$, $\frac{5}{1}$, $\frac{7}{1}$, $\frac{9}{1}$, etc.) and invert the various fractions, we have a harmonic series, as $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, etc. In any of these series, any member is a mean between its left-hand and its right-hand neighbors. For instance, in the harmonic series above, $\frac{1}{5}$ is the mean between $\frac{1}{3}$ and $\frac{1}{7}$. It needs only to be added here that between the same two quantities the arithmetic mean is numerically greater than the geometric, and the geometric is greater than the harmonic. For instance, between 4 and 25 the arithmetic mean is $14\frac{1}{2}$; the geometric, 10; the harmonic, $6\frac{26}{29}$. The three series are:

4,	$14\frac{1}{2}$,	25	(arithmetic)
4,	10,	25	(geometric)
4,	$6\frac{26}{29}$,	25	(harmonic)

The conception of average is so wrought into the consciousness from early life that it scarcely needs elucidation. Perhaps it should be said rather that it would need no elucidation, so far as this discussion goes, had not Jevons succeeded in enveloping it in a perplexing maze of uncertainty. One or two examples of averages, whose mere statement will compel assent, will suffice for our purpose. If two boys are aged, respectively, 6 and 10 years, their average age is 8 years, the arithmetic mean. $6+10=16$, and $8+8=16$. That is, the conception of an average is such that if the number of units in it be substituted in each term of the series to be averaged, the aggregate number of units is not changed. If a quantity fails in this test it must be discarded—it is not an average.[1] If two clocks are, respectively, 20 and 30 minutes fast, they average 25 minutes fast, the arithmetic mean. The last example may be put thus: If two clocks vary from the true time, respectively, by 20 and 30 minutes, their average variation is 25 minutes, the arithmetic mean. Why should we not say of two prices, say of wheat, if the price of one bushel is $1.50, and the price of another bushel[2] is $2.00,

[1] To the objection that this conception of an average is arbitrary, the reply is (1) that it conforms to ordinary usage, and (2) that no other conception is available to determine the price movement. This latter consideration, which has special importance in this discussion, is developed in the text. The movement of price can be determined by a comparison of ratios—the ratios of total prices, or values, as compared with the ratios of total quantities. The alternative and convenient method is by the use of averages, and it goes without saying that the two methods should tally in their results.

[2] It may not be amiss to caution against taking the simple average between the unit prices, in case of unequal quantities of a commodity. If 2 bushels of wheat sell at $1.00 per bushel and 3 bushels sell at $1.50 per bushel, the average price is not $1.25, but $1.30 per bushel, as follows:

2 bushels at $1.00 per bushel bring $2.00
3 bushels at $1.50 per bushel bring $4.50
Total, 5 bushels at a total of $6.50 is $1.30 per bushel.

The same result, of course, may be obtained by distributing the units of commodity with their prices, and taking the arithmetic average, as follows:

1 bushel at the rate of $1.00
1 bushel at the rate of 1.00
1 bushel at the rate of 1.50
1 bushel at the rate of 1.50
1 bushel at the rate of 1.50
Total, 5 bushels, at a total of $6.50, or $1.30 per bushel.

In Jevons's work, unequal quantities are scarcely considered. Whatever the relative quantities sold, the price of the unit is taken, and the computation based on the average price for the two bushels is $1.75, the arithmetic mean? Or, if $1.00 be taken as the standard price for wheat, and we wish to get the average variation for two bushels sold, respectively, at $1.50 and $2.00, why will not the transparency of the following process vouch for the correctness of the result:

Bushel 1 varies from the standard price $0.50; bushel 2 varies from the standard price $1.00; hence the average variation is $0.75, or, adding $0.75 to the standard price, $1.00, the average price for the two bushels is $1.75 per bushel? If we express the relation of the varying prices by ratios or percentages of the standard price, the result is identical:

If bushel 1 sells for 200 per cent. of the standard price, and bushel 2 sells for 150 per cent. of the standard price, the average percentage for the two bushels is 175; or, if $1.00 be the standard, the average price for the two bushels is $1.75 per bushel.

In presenting the logic of his method for determining the average variation (see below), Mr. Jevons clouds the discussion by introducing two commodities.[1] So far as the particular point at issue is concerned it matters not whether we consider the variations applying to two periods and a single commodity, or those pertaining to two commodities and a single period. There are points of distinction affecting these two cases that will be taken up later; but at present, for the sake of simplicity, we will regard the two cases as one. Mr. Jevons's account of his selection of the geometric mean is as follows:

Thus the price of cocoa has nearly doubled since 1845–1850. It has increased by 100 per cent., so that its variation is now expressed by the number 200. Cloves, on the contrary, have fallen 50 per cent., and are now at

that. The question of unequal quantities is one that will be considered in connection with the principle of "weighting," and it need not encumber the present discussion.

[1] The propriety or significance of averaging the prices, or the variations of prices, of commodities so diverse as yards of cotton and pounds of tobacco will be discussed in another place. Mr. Jevons takes the affirmative as to variations, but the question of the proper method of ascertaining an average in no wise depends on any concrete significance of the figures or numbers involved. Whether these latter represent real prices or fictitious or hypothetical prices, or erroneous variations, we are not now concerned to discuss.

50. The arithmetic mean of these ratios would be ½ (200+50) or 125. The average rise of cocoa and cloves would then appear to be 25 per cent. But this is totally erroneous. The geometric mean of the ratios expressed by the numbers 200 and 50 is 100. On the average of cocoa and cloves there has been no alteration of price whatever. In other words, the price of one is doubled, of the other halved — one is multiplied by two, one divided by two — on the average, then, the prices of these articles remain as they were, instead of rising 25 per cent.[1]

Now two methods of obtaining an average that yield diverse results cannot both be right. If one is right, the other must be wrong. Moreover the discrepancy between the results would have to be quite insignificant to justify the adoption of the inaccurate method on the ground of its greater simplicity. But Mr. Jevons accentuates the divergency of results and thus emphasizes the importance of accuracy in method. The plausibility of the reasoning quoted above arises from setting the two ratios 2 and ½ face to face, to offset each other. But the ratio 2 in the case cited corresponds to an advance of 100 points, while ½ corresponds to a decline of but 50 points. And if we contrast 100 with 50 it is certainly equally plausible that the former more than offsets the latter, so that the average should show a variation from the original prices.

It is not difficult to expose the speciousness of Mr. Jevons's reasoning. Division and multiplication are reverse processes, and it sounds plausible to say that multiplying by 2 and dividing by 2 are mutually neutralizing operations, but it depends entirely on the quantities operated on. If 100 be multiplied by 2, and then the product 200 be divided by 2, the original quantity 100 is regained; but if we perform the two operations on the same quantity, as does Mr. Jevons, the result is not so simple. He multiplies 100 by 2 and divides 100 by 2, and then by dwelling on the identity of multiplier and divisor, and ignoring the remaining elements of the problem, he reaches a chimerical conclusion. This sort of reasoning will not stand a practical test for one moment. If a man should make two investments of $100 each, and realize 200 per cent. on one investment and 50

[1] *Investigations in Currency and Finance*, p. 23.

per cent. on the other, Mr. Jevons's style of reasoning would figure out no reward for the investor's pains. Modern book-keeping shows no such sterility in real transactions. The investor makes $100 on one transaction and loses $50 on the other, showing a net gain of $50.

Mr. Jevons adhered to the geometric mean in spite of adverse criticism, although he conceded some strength to the opposition, witness the following passage:

The reasons for adopting the geometric mean were explained in my pamphlet, and I still think those reasons sufficient. I must mention, however, that the method has been called in question by Dr. E. Laspeyres. . . . . Dr. Laspeyres urges, if I read him aright, that as the value of gold meant its *purchasing power*, we ought to take the simple arithmetic average of the quantities of gold necessary for purchasing uniform quantities of given commodities.[1] There is certainly some ground for the argument. But it may be urged with equal reason that we should suppose a certain uniform quantity of gold to be expended in equal portions in the purchase of certain commodities, and that we ought to take the average quantity. This might be ascertained by taking the *harmonic mean*. Thus there are no less than three different kinds of averages which might be drawn.[2]

Mr. Jevons does not venture an explanation of how a harmonic mean is obtained under the stated conditions, but contents himself with mathematical illustrations of the three kinds of *mean*, and by means of an example educes the result that "the mean rise of price might be thus variously stated:

	Per cent.
Arithmetic mean - - - - - - -	50
Geometric mean - - - - - - -	41
Harmonic mean - - - - - - -	33

It is probable that each of these is right for its own purposes when these are more clearly understood in theory." Then follow three remarkable reasons for adhering to his method:

Because (1) it lies between the other two; (2) it presents facilities for the calculation and correction of results by the continual use of logarithms, without which the inquiry could hardly be undertaken; (3) it seems likely to give in the most accurate manner such general change in prices as is due

[1] *Hildebrand's Jahrbücher*, vol. iii. p. 97.

[2] *Ibid.*, p. 120.

to a change on the part of gold. For any change in gold will affect all prices in an equal ratio, etc.

The first two of these reasons are remarkable from Jevons's own original standpoint of the importance of accuracy. The third reason is remarkable in its contention, since a change on the part of gold is manifested in the same way as a change on the part of commodities, namely, in the prices; and the simple question of the average of prices, or the average of variations in prices, is not affected by the causes of those variations. If so, we might need a different method of computation for every cause of variation.

That the arithmetic mean gives the correct average will be obvious if the relation of price is stated in expanded form in terms of purchase. For instance, if one dollar purchase one bushel of wheat at one date, and one dollar at another date purchase two bushels, the two dollars purchase in the aggregate three bushels, which yield an average purchase of one and one half bushels for one dollar, or the arithmetic mean between one bushel and two bushels. Let us now describe what takes place if we reverse the terms of the problem. If one bushel purchase one dollar today, and one bushel purchase two dollars tomorrow, two bushels purchase in the aggregate three dollars, or one bushel on the average has a purchasing power of one and one half dollars, the arithmetic mean. This latter statement reflects with simple fidelity the mathematical relation involved in price changes, and establishes beyond cavil the correctness of the arithmetic method. The same result follows if we consider variations of price from a given standard instead of the prices themselves. For instance, if one dollar per bushel be taken as the standard price for wheat, and we wish to ascertain the average variation between two dollars per bushel and one half dollar per bushel, for which Mr. Jevons's method gives as the result no variation, the correct result is evident from the following process:

At two dollars, the excess for one bushel is -	$1.00
At one half dollar, the deficit for one bushel is -	50
The aggregate excess for the two bushels is - -	50

which gives an average excess of .25 to the bushel, showing the average price of the wheat to be $1.25, or .25 in excess of the standard taken.

Mr. Jevons intimates that a certain method of looking at the problem will yield a process that results in a "harmonic mean." As "harmonic mean" is indissolubly associated with "arithmetic mean," by definition, it follows that the harmonic mean can be made to emerge wherever the arithmetic mean figures in a result; but in this case the former certainly does not emerge as the correct register of an average. To illustrate:

(1) Given, $1 purchases 1 bu.
(2) Given, $1 purchases 2 bu.
(3) Result, $1 purchases 1 ½ bu. (Average).

1, 1½, 2 are in arithmetical progression, and if we compute the purchasing power of 1 bushel of wheat in terms of money at the three different rates, we get a harmonic progression, as follows:

At $1 per bu., 1 bu. purchases $1
At $1 per 2 bu., 1 bu. purchases $½
At $1 per 1 ½ bu., 1 bu. purchases $⅔

1, ⅔, ½ are in harmonic progression, ⅔ being the harmonic mean between 1 and ½. This harmonic mean does not express an average. It is evident that the average price of one bushel of wheat at the three rates of

1 bu. purchasing $1
1 bu. purchasing $½
1 bu. purchasing $⅔

is ascertainable by summing up the second column and dividing by 3, which is an *arithmetical* process. The harmonic mean, ⅔, in the above case is obtained by first forming an arithmetical series, the middle term of which denotes an average, and then, owing to the inevitable reciprocity of the problem, the conditions furnish the harmonic series that is inseparably bound up with the former. The arithmetic series is 1, 1½, 2, or $\frac{1}{1}$, $\frac{3}{2}$, $\frac{2}{1}$. Inverting each of these fractions we have 1, ⅔, ½, a harmonic series by definition. The harmonic series is bound up with the arithmetic series by the reciprocal relation of commodity and price; or, to speak more accurately, by the reciprocal relation of the units that express quantitatively the commodity on the one hand and the standard of value on the other.

A writer in the *Quarterly Journal of Economics* for October 1886, seems to think that the character of an average, when pressed into the service of political economy, is so thoroughly fictitious that it is quite optional to employ one mean or another as fancy or circumstances may dictate.[1] If this were true, the labor of compiling tables involving averages would be indeed a useless and fruitless task. Fortunately, it is not true. This writer bungles distinctions. It is true that the average price does not register a concrete realty, but as a mathematical relation existing among *real* prices, it is as real and definite as the prices themselves. This means that fictitious averages are simply pseudo-averages, and result from erroneous computation.

It will be interesting to note into what a tangled web this notion of a fictitious average leads. As a notion, it is the creation of Jevons presented in his *Principles of Science*, chap. XVI. Its style of service may be briefly illustrated. Consider two commodities, one of which has remained stationary, or at 100 per cent., and the other has doubled in price, or advanced to 200 per cent. The arithmetic average is 150 per cent., or 50 per cent. advance. Now this number 150 marks a deviation of 50 from both 100 and 200; it therefore embodies an error of 50, when made to represent the one or the other. This saddles a greater proportional error on the smaller number, but as the choice of a fictitious average is under no sort of constraint, one has a right to favor the larger quantity. However, if the quantities are regarded equally important, the choice of the average would naturally fall on the harmonic mean. In the example cited,

[1] "Its fictitious character renders it possible to make choice among different values, and thus among different methods of finding it. This is generally overlooked by those who invariably use the arithmetic mean as if it were the only one which could be applicable. The only justification for any fictitious mean is to be found in its convenience as a representative of the true quantities. It is upon this criterion that Mr. Jevons based his choice."—F. COGGESHALL, "The Arithmetic, Geometric, and Harmonic Means."

133⅓ is the harmonic mean between 100 and 200, giving an error of 33⅓ and 66⅔, respectively, for the two quantities averaged, evidently an equitable apportionment, as 33⅓ and 66⅔ are, respectively, one third of 100 and 200.

The fanciful character of this conception of an error that requires distribution will appear, if we introduce some prices between these two extremes, 100 and 200, such as will not disturb the average. Suppose a third commodity has advanced to 133⅓ per cent., which will not disturb the average. Such a commodity would get no share of the error, and yet, according to the reasoning, this commodity is much better able to stand it than the one represented by 100 per cent. It is clear that if a number of commodities are to share an error in proportion to ability to stand it, the average cannot exceed the price represented by the minimum percentage in the scale of variation. For instance, if several commodities suffer variations from standard price, represented by 50, 51, 75, 125, 200 per cent. and the average must be taken so as to distribute the error according to the magnitude of these numbers, this average must be taken less than 50½. Now, if we introduce commodities whose scale of variation will fall between 50 and 50½, it is plain that our logic will push the average down to 50. Of course any figure lower than 50 will answer the same purpose.

The absurdity of the geometric method will be manifest if we consider cases of extreme variation. To begin with Mr. Jevons's illustration, a rise expressed by the ratio 2 is offset by a decline expressed by ½. In the same way a rise expressed by 4 is offset by a decline expressed by ¼; or, which is the same thing, if two commodities each double in price, their variation is offset by that of a single commodity declining to ¼ its standard price. By the same logic 100 commodities doubling in price would be offset by one commodity declining to $\frac{1}{2^{100}}$ its standard price. That is to say, one commodity by declining in price can offset any rise in all other commodities combined, and the result will be an average of no variation.

We can bring out the absurdity in a still bolder form. Let us take the case of a commodity that may have a compass of price, including zero, such as water. By the geometric method what is the average price of water between its extreme rates, taking the maximum rate, 2, and the minimum rate 0? The true average is the arithmetic mean, 1. The geometric mean is $\sqrt{2 \times 0} = 0$. That is, the average price of water by the geometric method would be its minimum price. The harmonic mean in this instance is the same as the geometric mean, and hence is equally absurd as representing an average. Indeed, the introduction of the harmonic mean can serve no other purpose than to cloud the transparency of a simple problem, and thereby furnish the pretext for a compromise between two so-called equally plausible results.

Professor F. G. Edgeworth has entered the lists with a novel suggestion for an average. Representing the variations in the prices of the various commodities by percentages, he writes these percentages down in the order of their magnitude, and selects the central member of the series as the average. For instance, if there are five commodities, and the percentages representing their variation are 75, 90, 95, 115, 125, the average variation is 95. Professor Edgeworth styles this mean the *median*. Its claims are earnestly urged in the Reports of the British Association for the Advancement of Science, 1888, 1889, in the *Journal of the Royal Statistical Society*, June 1888, and elsewhere.

The method of the median has the conspicuous merit of extreme simplicity, *conspicuous* because its *sole* merit. It is recommended by Professor Edgeworth to serve for a special sort of cases which he calls typical. "For the purpose of a mere average or type, we are to take account of all manner of goods, and we are not concerned with the quantity of each commodity. We have for this purpose only to ascertain the ratios or percentages . . . . and then to take a simple or unweighted mean of these ratios."[1]

[1] "Appreciation of Gold," *Quarterly Journal of Economics*, January 1889, p. 161.

This is the type of cases in which the method of the median is especially appropriate, according to Edgeworth. On the contrary, this is the type of cases that affords no pretext for deviating from the true or arithmetic method. Where data are deficient, it is possible that certain devices may furnish an approximate average, but in Edgeworth's typical cases, the data are all at hand, quantities are not considered—that is to say, the quantities are taken uniform for all commodities. In these cases the arithmetic method is eminently practicable as well as theoretically appropriate.

Professor Edgeworth is willing to indulge any prejudice in favor of weighting the percentages. "However, it may be admitted," he says, "that though there is no peculiar propriety in using a *weighted* mean for the present purpose, at the same time there is not much harm in doing so."[1] In another paper he makes the concession stronger. "There is a variety [of median] constituted by assigning special importance to those returns which we have reason to suppose are specially good representatives of the changes affecting the value of money."[2] Then follow proposals for weighting, according to one or two simple devices, which strike one as jocular rather than serious.

As Professor Edgeworth's median logically contemplates at best but an approximate result, it need detain us no longer.

II. THE DEFECT IN INDEX NUMBERS AND THE REMEDY.

The initial and simplest problem connected with prices or index numbers is to ascertain the comparative prices of a single commodity for successive periods. For convenience a period of comparative stability of the market is taken as the base period, and the prices of the various periods under consideration are compared with the average price for this period. If index numbers are used, the price of the base period is represented by 100, and all prices are reduced to the scale of this base. For instance, consider some specific commodity, as oats. If oats are rated at

[1] *Ibid.*, p. 162.

[2] *Report of the British Association for the Advancement of Science*, 1888, p. 207.

40c. per bushel during the standard period and rise to 50c. at a subsequent period, the index numbers for the two periods are 100 and 125. The important point here pertains to the method of ascertaining this price of 40c. or 50c. Of course, the figures denote the average price, but precisely how is this average computed? It is customary to take prices at stated times during the period, once a quarter, or once a month, and compute the simple average of the schedule thus obtained. This will usually answer for practical purposes, especially if the variation during the period is slight; but a higher degree of accuracy will be obtained by allowing for the differences of quantity sold at the different rates. If twice the quantity is sold at one rate as compared with another, the former rate should be taken twice in determining the average. To make the matter perfectly clear, take the following schedule for oats for a year by the month: January, 40c.; February, 50c.; March, 55c.; April, 45c.; May, 40c.; June, 40c.; July, 50c.; August, 55c.; September, 45c.; October, 35c.; November, 40c.; December, 45c. The simple average computed by adding the various figures and dividing by 12 is 45c., which ordinarily would be taken as the average. For the sake of simplicity of results we will suppose that equal quantities are sold by the month with the exception of October, which schedules the lowest price. It would not be a singular phenomenon if the low price should attract an unusual sale. If the quantity sold in October should be thirteen times as great as that for any other month, the price 35c. for that month should figure thirteen times in the sum, and the divisor should be 24 instead of 12, giving the result 40c., or 5c. less than the former result. This gives an importance to the price 35c. thirteen times as great as that for any other month due to the proportionally larger quantity for October. It is not usual to speak of "weighting" when considering a single commodity; but the propriety of so doing is perfect, as is evident from the above example, and the significance is clear. Strictly construing an average, it is the result of dividing the total proceeds of a commodity during the period, say a year, by the number of units of the commodity involved in the transactions. If three million bushels and 1.2 million dollars be these totals, the latter divided by three million gives 40c. as the average price per bushel. The same result would follow, if the prices scheduled by the month were weighted proportionally to quantities sold, and the weighted figures entered into the calculation.

When we come to a plurality of commodities, the case is not quite so simple. Indeed the subject of averages and index numbers as applied to a variety of commodities has been a theme of endless controversy. Jevons justified the propriety of averaging *variations* in distinction from the prices themselves as follows:

> There is no such thing as an average of prices at any one time. If a ton of bar-iron costs £6, and a quarter of corn £3, there is no such relation or similarity between a ton of iron and a quarter of corn as can warrant us in drawing an average between £6 and £3; and similarly of other commodities. If at a subsequent time a ton of iron costs £9, and a quarter of corn £3 12s., there is again no average between these quantities. We may, however, say that iron has risen in price 50 per cent. or by $\frac{1}{2}$; what was previously 100 has become 150; corn has risen 20 per cent. or by $\frac{1}{5}$; what was 100 has become 120. Now the ratios 100: 150 and 100: 120 are things of the same kind, but of different amounts, between which we can take an average.[1]

The above statement is so far erroneous that almost the exact reverse is true. It is a very simple matter to take the average price of any number of thinks alike or unlike, as is evident from the conception of an average; but in general it will be found impossible to take an average between variations.

But first let us examine some of the methods of averaging that have obtained more or less sanction and prestige. Jevons used, generally speaking, unweighted prices or index numbers. That is, if the index numbers for a variety of commodities for any period were found to be, say, 90, 105, 99, the average index number would be simply the sum divided by 3. Most statisticians at present reject this sort of an average. They insist that the prices, for instance, of salt, mercury, wheat and beef cannot be summed up and divided by 4, with any significant result.

[1] *Investigations in Currency and Finance*, p. 23.

Professor Edgeworth, however, approves this very process as applicable to one of his species of averages:

For the purpose of a mere average or type, we are to take account of all manner of goods, and we are not concerned with the quantity of each commodity. We have for this purpose only to ascertain the ratios or percentages, such as

$$\frac{\text{Present price of anchovies}}{\text{Original price of anchovies}} \qquad \frac{\text{Present price of alkali}}{\text{Original price of alkali}} \qquad \frac{\text{Present price of beef}}{\text{Original price of beef}}$$

and then take a simple or unweighted mean of these ratios. . . . . This rule will excite the mirth of some. What, they will say, assign the same importance to pepper and nutmeg as cotton and iron! Yes, I reply, for the present purpose, etc.[1]

If we could ignore quantity our problem would be much simplified; but the plain truth is, that if we do ignore quantity, we vitiate our results. The simple average of the prices of pepper, nutmeg, cotton and iron cannot be taken with any significant result, for the reason that such an average would not correspond to the phenomena of actual facts. Such an average price could not be applied to each of the four commodities, to get the total proceeds from the real transactions of any period of time under consideration; and an average price has no meaning unless it satisfies this condition. The simple average as taken by Professor Edgeworth can apply only to a hypothetical case. If we suppose a unit of each of the commodities to be sold at a specific rate, the average rate for these several transactions is the simple average; but such a case affords no instruction for the problem engaging us.

This point is so important in our discussion that it will warrant a simple and careful presentation. We have seen[2] that in the case of a single commodity, an average has to do essentially with quantities. If a bushel of wheat sells for $1.00, and a second bushel for $1.50, the average for the two bushels (quantity)

[1] "Appreciation of Gold," *Quarterly Journal of Economics*, January 1889, pp. 161–162.

[2] See footnote, p. 173.

is $1.25. Now if two bushels sell at $1.00 per bushel, and three bushels sell at $1.50 per bushel, the average price is not $1.25. This would ignore quantity. If $1.25 is substituted for the two rates, $1.00 and $1.50, the five bushels aggregate $6.25, whereas in the real transaction we get

2 bushels at	$1.00	per bushel,	=	$2.00
3 "	$1.50	"	=	4.50

or five bushels aggregate $6.50. The true average price is $6.50 divided by 5 (quantity), or $1.30.

Now it would be passing strange if the factor of quantity, so indispensable in the case of a single commodity, could be ignored when a plurality of commodities is concerned. The truth is that it is this very ignoring of quantity (the ignoring altogether, or the inadequate providing for quantity by recourse to arbitrary systems of weighting) that has led to divergent and unsatisfactory results.

The meaning will be clear by taking a simple example. If the average prices of wheat and rye are $1.00 and .50 for a specific period, it will not do to take the simple average of $1.00 and .50 (or .75) for the average price of the two commodities. Suppose the transactions in wheat for the period are four million bushels, for rye one million bushels; we have

4,000,000	bushels at	$1.00	per bushel,	=	$4,000,000
1,000,000	"	.50	"	=	500,000
or 5,000,000	" at a total of				4,500,000
Now 5,000,000	" at .75 per bushel			=	3,750,000

proving the inadequacy of the simple average, .75. The true average is obtained by dividing the total proceeds, 4.5 million dollars by the total number of bushels involved, five million (quantity), which yields .90 as the correct result. Ninety may be substituted for the prices of wheat and rye without affecting the totals, the decisive test of a correct average.

In the above example of wheat and rye, but one species of unit is involved, the bushel. Let us now consider the case involving different units of quantity. Let us see whether there is any significance to an average price between bar iron and corn.

Jevons says there is not. "If a ton of bar iron costs £6, and a quarter of corn £3, there is no such relation or similarity between a ton of iron and a quarter of corn as can warrant us in drawing an average between £6 and £3." Immediately, however, as quoted a few pages back, he goes on to show that we can draw an average between their *variations*, because variations may be represented as percentages or ratios, and ratios "are things of the same kind . . . . between which we can take an average."

Is not price a homogeneous relation the same as ratio, and susceptible to the same sort of comparison and manipulation? Has ratio a superior sort of homogeneity? It is difficult to see wherein. The fact is that homogeneity is not a sufficient criterion here. We are considering relations, and Jevons draws an average between relations, neglecting meantime the objects related, and then he applies the result to the objects. It has been shown that such a proceeding is invalid in case of prices, and price has a *primary* relation to the object. Quantity must be taken account of. Now the ratios referred to here are primarily relations among the prices, and hence stand in a *secondary* relation to the objects themselves; so that the application of averages of ratios to the objects must be a more precarious matter than in the case of the immediate relation of price. It will appear later that ratios, homogeneous as they are, have to be handled with discriminating recognition of quantity to reach an intelligent interpretaion of results.

With the *prices* themselves for a single period the case is different. If the price of iron is £6 per ton, and that of corn £3 per quarter, we may get the average price, but first we must take into account the quantity of each commodity. Say the volume of business in iron for a specific period is 100,000 tons; for corn 200,000 quarters. If the average prices for the two commodities are £6 and £3, we have

100,000 tons of iron at £6, = £600,000
200,000 quarters corn at 3, = 600,000

Total, 300,000 units of the two commodities aggregate £1,200,-000, or £4 per unit as the average for the two commodities.

That this is the true average may be proved by substituting it in the computation as follows :

100,000 tons at £4	=	£400,000
200,000 quarters at £4	=	800.000
Total,		£1,200,000

as above.

If it be objected that it is absurd to speak of £4 as applied to a ton of iron, since it may be that iron never sold as low as that figure, the reply is that £4 does not purport to be the price of iron; it as an average price taken between two commodities. One might with equal propriety object to averaging the prices of two grades of corn on the ground that the better grade never sold as low as the average. The confusion arises from not keeping in mind the significance of the concept, average, which is nothing more or less than a vicarious quantity, which, when substituted for the quantities that it replaces, does not change the totality of units comprehended in these various replaced quantities. In the above case, the average £4, substituted for £6 and £3, does not increase or diminish the totality (£1,200,000) of units contained in the quantities, £6 and £3, taken the proper number of times, namely, 100,000 and 200,000 respectively.

We must now establish the *nexus* between price schedules and index numbers. The convenience and attractiveness of the latter are such that the incentive to use them in price statistics is great. On the face of it, there is nothing simpler and more logical than the subsitution of a convenient scale of ratios for a schedule of prices. For instance, 100, 80, 125, 200 may be substituted for 20, 16, 25, 40, and the former schedule operated on by all the processes of division, multiplication, etc., involved in the taking of averages, and the results applied to the latter schedule, with complete confidence in the trustworthiness of the conclusions. All that is necessary in the above case is to divide every result obtained from the former schedule by 5. This is evidently valid, since 100, as representing a price, may stand for the same *value* as 20. If 20 denote that number of *cent* units, changing 20 to 100 simply reduces the unit, *cent*, to *one fifth of a cent*. In the one case, we have 20 *cents;* in the other case, we have 100 *one fifth cents*. The numbers, 100, 80, 125, 200 must yield as trustworthy results as the proportional numbers, 20, 16, 25, 40. In fact, as shown above, they must yield identical results.

And yet the use of index numbers has been condemned on the ground that they give inaccurate results. M. G. Mulhall[1] pronounces a sufficiently sweeping condemnation which is equally appropriate as against the genuine schedule of prices. He takes it for granted that index numbers mean unweighted numbers. If the proper weights be applied, his objections lose their force. A more serious objection charges essential erroneousness to the principles of construction of index numbers. N. G. Pierson[2] and C. W. Oker[3] urge in strong terms the untrustworthiness of this instrumentality for measuring price variations. Their modes of presenting the subject are sufficiently distinct, but the principle in the two cases is the same, so that a single examination will suffice. Pierson's illustrations accentuate the discrepancies in the results more strongly than those of Oker. We quote from the former:

Let us suppose ten commodities, all equally important. Five of them are doubled in price, and five of them fall exactly to one half. Supposing these ten commodities to have been equally cheap or dear before the changes occurred, it is evident that their average price will have risen 25 per cent.

First period	Second period
5 × 100 = 500	5 × 200 = 1000
5 × 100 = 500	5 × 50 = 250
10)1000	10)1250
100	125

The index numbers in this table give a correct account of the alteration which has taken place; they show a rise from 100 to 125.

[1] *History of Prices*, 1885, p. 7.

[2] *Economic Journal*, March 1896.

[3] JOURNAL OF POLITICAL ECONOMY, September 1896.

But if we had started from the *second* period, expressing the initial prices in percentages of the prices as they were after the change, we should have found something quite different.

First period	Second period
5 × 50 = 250	5 × 100 = 500
5 × 200 = 1000	5 × 100 = 500
10)1250	10)1000
125	100

In this case the index numbers would have shown a fall instead of a rise.

The above criticism chronicles a fatal objection to index numbers, as they have been constructed. Professor Edgeworth,[1] in the same number of the *Journal*, undertakes to meet the objection; but the "Defense" is not based on a denial of Pierson's theoretical contention, but in the fact that the latter's illustrations are not typical; they are extreme hypothetical cases, created to magnify what in actual experience is an insignificant source of discrepancy.

To sum up, several of Mr. Pierson's objections amount to this one: that the calculation of average variations in prices is untrustworthy because the result is seriously different according as different systems of weighting are employed. And the objection, though true in the abstract of artificially simplified index numbers, is not true of the sets of figures with which we have actually to deal.[2]

Edgeworth's reply is inadequate. The objection that Pierson raises is real. The system of index numbers as constructed and used has a fundamental imperfection that vitiates the results. A slight change, however, will remedy the defect, and will enable the system to secure absolutely accurate and uniformly consistent results. Mr. Pierson himself trod on the very heels of this remedy, but it managed to elude his grasp. In speaking of his illustrations, he says:

The first table shows the effect of variations in the value of commodities of which prices were originally *equal:* the second table applies to commodities of which prices were originally *unequal.* Each percentage of a high price has a greater arithmetical importance than each percentage of a low one. But the system of index numbers takes no account of this difference.[3]

[1] "A Defense of Index Numbers."

[2] *Economic Journal*, March 1896, p. 136.

[3] P. 128.

This last sentence contains the key to the situation. Index numbers *should* take account of this difference. The defect in them springs out of this omission. It is futile to take unequal prices and deal with them as though they were equal, expecting the laws of mathematics to condone the offense. But this is just what the present system of index numbers does. It represents \$5 and \$50 and .05 indiscriminately by 100, without even an apology for the unwarranted proceeding. Now it is plain, in case of two commodities, one priced at .05 and the other at \$50, that if the former advances 100 per cent., making its price .10, and the latter remains stationary, there is not an average advance of 50 per cent., not certainly if a unit only of each commodity is considered, or the same number of units. But index numbers bring about this fantastic result as follows:

For the .05 commodity, $200 \times 1 = 200$
For the \$50 commodity, $100 \times 1 = 100$

$$2 \overline{)300}$$
$$150$$

or 50 per cent. average increase.

The trouble comes from representing each commodity by 100 without making compensation. How different the result if we make proper compensation. The \$50 commodity and the .05 commodity may be represented by the same number 100; but we must remember that in so doing we virtually change the unit of quantity without allowing for the change, .05 being $\frac{1}{1000}$ of \$50, if we represent the price of the \$50 commodity by 100, we may take the same number 100 to represent the other commodity, *only* if we at the same time increase the unit of the latter to one thousandfold. The price of this thousandfold unit will then be \$50, the same as that of the first commodity. However, as we had but *one* of the smaller units, we have now to consider $\frac{1}{1000}$ of the thousand-fold unit, and our table stands thus,

For the .05 commodity, $200 \times \frac{1}{1000} = .2$
For the \$50 commodity, $100 \times 1 = 100.0$

$$1.001 \overline{)100.2000}$$
$$100.1—$$

that is, the average advance for the two is not quite .1 per cent. instead of 50 per cent.

Mr. Pierson's first table gives the correct result, because he stipulates that the articles have the same initial price, so that he is justified in assigning the index number 100 to each commodity or each group. In his second table he takes the second period as the base period, but, the prices having diverged, he is not warranted in assigning the same index number to each group without making the proper compensation. We can easily readjust this table, so as to show the same average variation as the first table. The price of the first group of commodities in the second table being four times that of the second, we may consider a fourfold unit of each of this second group, and represent the price of this new unit identically with that of the first group. We now have to consider $\frac{5}{4}$ units of the second group ($\frac{1}{4}$ unit of each member of this group) with the following result:

1st period	2d period
$5 \times 50 = 250.0$	$5 \times 100 = 500$
$\frac{5}{4} \times 200 = 250.0$	$\frac{5}{4} \times 100 = 125$
$6.25\overline{)500.00}$	$6.25\overline{)625.00}$
80	100

This table shows the same average advance, 25 per cent., as the first table of Mr. Pierson, and is the true record of the variation.

It is plain that the trouble with index numbers arises from subsituting for a set of miscellaneous prices one uniform number 100, instead of using proportionals throughout, as the inflexibility of mathematical principles enjoins. Statisticians, however, have considered it convenient to use one uniform base number, 100; and mathematics is sufficiently indulgent to permit this innovation also, but she demands rigorous compensation, as outlined above.

It will be noticed in the examples cited that a correct system of index numbers requires a strict recognition of quantity, in order to secure accurate results. Quantity is essential whether we use the schedule of actual prices or substitute index numbers.

Mr. Pierson entirely loses sight of this imperious necessity, and, in the latter part of his article, he turns hopeless pessimist on the question of computing average variations of prices by any method whatever. It would be possible, he grants, if we had one uniform standard of measure, but the inevitable variety leads to inextricable confusion and despair. By changing the unit from one of weight to bulk, Mr. Pierson finds that the transposition evokes a veritable wizard who changes plus into minus, and, by a subtle magic, transforms increase into decrease. The process is as follows:

Now let us suppose that—

100 pounds of A are equal to 1 bushel
100 pounds of B are equal to 0.5 bushel
100 pounds of C are equal to 2 bushels

Then, if each of the three articles is worth 20s. per hundred pounds, they will be worth per *bushel*,

A	20s.
B	40s.
C	10s.

Supposing commodity A to rise 25 per cent., B to fall 50 per cent., and C to rise 50 per cent., this would affect the average price as follows:

If the price is expressed per 100 pounds,

A will rise from 20 to 25s.
B will fall from 20 to 10s.
C will rise from 20 to 30s.

60 to 65s.

Which means an average rise from 100 to 108.3s. But if the price is expressed per bushel,

A will rise from 20 to 25s.
B will fall from 40 to 20s.
C will rise from 10 to 15s.

70 to 60s.

Which means an average fall from 100 to 85.7.

Thus it would simply depend on the method of expressing the price—per 100 pounds or per bushel—whether an average rise or an average fall were recorded.

I do not see my way out of this difficulty and the only possible conclusion seems to be that all attempts to calculate and represent average movements of prices, either by index numbers or otherwise, ought to be abandoned.[1]

[1] P. 131.

This trick of transposition from pounds into bushels, by which an advance is changed into a decline, is easily exposed. It is accomplished by an unavowed substitution of quantities. With the figures showing the advance from 60 to 65s., 100 pounds of each commodity are taken. When reduced to bushels the quantities vary, but Mr. Pierson, for the sake of having the quantities uniform, deliberately and naïvely makes the change without realizing its significance. Let us see the result when we adhere to the original quantities expressed in the new standard:

1	bushel of A rises from	20 to 25s.
0.5	bushel of B falls from	20 to 10s.
2	bushels of C rises from	20 to 30s.
		60 to 65s.

or, of course, the identical result that Mr. Pierson finds in the first instance. This amounts simply to weighting for quantity. The quantities being 100 pounds for each commodity, when we change to bushels, we must weight A with thc coefficient 1, B with ½, C with 2, and the transaction may be expressed as follows:

	Coef.	Importance.		Price.		
A -	1 ×	1 = 1.0	@	20s. = 20s.;	25% advance gives	25s.
B -	- 1 ×	½ = 0.5	@	40s. = 20s.;	50% decline gives	10s.
C -	1 ×	2 = 2.0	@	10s. = 20s.;	50% advance gives	30s.
				60s.	to	65s.

Mr. Pierson's trick can be performed without transforming into bushels by simply modifying the quantity of each article so as to express the equivalence of 1 bushel, and then ignore the quantitative discrepancy. Instead of 100 pounds of each article we have by this arrangement:

100 lbs. of A command 20s.;	25% advance gives	25s.
200 lbs. of B command 40s.;	50% decline gives	20s.
50 lbs. of C command 10s.;	50% advance gives	15s.
70s.	to	60s.

Of course, as thus performed, the trick is no trick at all, because the change of quantities is so obtrusive that no one is deceived.

III. THE EQUIVALENCE OF THE TWO METHODS, PRICES AND INDEX NUMBERS; THE CHARACTER AND LIMITATION OF THEIR SERVICE.

Index numbers properly constructed are as trustworthy as tables of actual prices. They are neither more nor less reliable. They give and must give proportionate results with the severe certainty of mathematical law. In case of single commodities, either method will indicate accurately the price relations among the various periods. Either method will give, for any isolated period, the correct average price of all commodities. In case the quantities of each commodity for the various periods are proportional, either method will give the correct general movement of price, giving the ratios of the average prices of all the commodities between period and period as accurately as in the case of single commodities. Both methods are powerless beyond this point. The bounds of their capacity are rigorously set at this limit. Neither method is competent when the quantities become disproportionate. We will establish these propositions by means of examples that are simple and at the same time decisive.

The first example, exhibited in Tables I to IV, involves proportional quantities. Table I is constructed of actual prices. Tables II, III, IV employ index numbers. The four tables present the same phenomena with harmonious results. Any table can be readily transformed into any other by the use of simple mathematical formulæ. The method and rationale of the changes have been already indicated. The last three tables take the first, second and third periods respectively, as base periods. To transform Table I, for instance, into IV, the third period is taken as the base period, and 100 represents the price of each commodity. As 90 ($0.90 may be read 90c., so as to avoid the complication of the decimal point) is the actual price of wheat in the third period, 100 may be regarded as the price of $\frac{10}{9}$ of a bushel; so that the relations of Table I will be preserved in IV if we change the unit of quantity from 1 bushel to $\frac{10}{9}$ of a bushel, and at the same time apply a compensatory ratio to the number of units, *i. e.*, take $\frac{9}{10}$ of 100, or 90, as the

TABLE I.

Commodity	Unit	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
Wheat	1 bu.	\$ .80	× 125 =	\$ 100.00	\$1.00	× 475 =	\$ 475.00	\$ .90	× 100 =	\$ 90.00	\$ .95	× 700 =	\$ 665.00
Sugar................	1 cwt.	4.00	× 200 =	800.00	5.00	× 760 =	3,800.00	6.00	× 160 =	960.00	$4.96\frac{3}{7}$	× 1,120 =	5,560.00
Wine................	1 gal.	5.00	× 50 =	250.00	4.00	× 190 =	760.00	3.00	× 40 =	120.00	$4.03\frac{4}{7}$	× 280 =	1,130.00
Totals			375	\$1,150.00		1,425	\$5,035.00		300	\$1,170.00		2,100	\$7,355.00
Average price of all commodities (Price of composite unit)				\$ $3.06\frac{2}{3}$			\$ $3.53\frac{1}{3}$			\$ 3.90			\$ $3.50\frac{5}{21}$

TABLE II.

Commodity	Unit	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
Wheat	1 bu.	100	× 125 =	12,500	125	× 475 =	59,375	$112\frac{1}{2}$	× 100 =	11,250	$118\frac{3}{4}$	× 700 =	83,125
Sugar................	$\frac{1}{5}$ cwt.	100	× 1,000 =	100,000	125	× 3,800 =	475,000	150	× 800 =	120,000	$124\frac{3}{28}$	× 5,600 =	695,000
Wine	$\frac{4}{25}$ gal.	100	× $312\frac{1}{2}$ =	31,250	80	× $1,187\frac{1}{2}$ =	95,000	60	× 250 =	15,000	$80\frac{5}{7}$	× 1,750 =	141,250
Totals			$1,437\frac{1}{2}$	143,750		$5,462\frac{1}{2}$	629,375		1,150	146,250		8,050	919,375
Average price of all commodities (Price of composite unit)				100			$115\frac{5}{23}$			$127\frac{4}{23}$			$114\frac{67}{322}$

TABLE III.

Commodity	Unit	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
Wheat	1 bu.	80	× 125 =	10,000	100	× 475 =	47,500	90	× 100 =	9,000	95	× 700 =	66,500
Sugar................	$\frac{1}{5}$ cwt.	80	× 1,000 =	80,000	100	× 3,800 =	380,000	120	× 800 =	96,000	$99\frac{2}{7}$	× 5,600 =	556,000
Wine	$\frac{1}{4}$ gal.	125	× 200 =	25,000	100	× 760 =	76,000	75	× 160 =	12,000	$100\frac{25}{28}$	× 1,120 =	113,000
Totals			1,325	115,000		5,035	503,500		1,060	117,000		7,420	735,500
Average price of all commodities (Price of composite unit)				$86\frac{42}{53}$			100			$110\frac{20}{53}$			$99\frac{46}{371}$

TABLE IV.

Commodity	Unit	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Price	Units (Quantity)	Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
Wheat	$\frac{10}{9}$ bu.	$88\frac{8}{9}$	× $112\frac{1}{2}$ =	10,000	$111\frac{1}{9}$	× $427\frac{1}{2}$ =	47,500	100	× 90 =	9,000	$105\frac{5}{9}$	× 630 =	66,500
Sugar................	$\frac{1}{6}$ cwt.	$66\frac{2}{3}$	× 1,200 =	80,000	$83\frac{1}{3}$	× 4,560 =	380,000	100	× 960 =	96,000	$82\frac{31}{42}$	× 6,720 =	556,000
Wine	$\frac{1}{5}$ gal.	$166\frac{2}{3}$	× 150 =	25,000	$133\frac{1}{3}$	× 570 =	76,000	100	× 120 =	12,000	$134\frac{11}{21}$	× 840 =	113,000
Totals			$1,462\frac{1}{2}$	115,000		$5,557\frac{1}{2}$	503,500		1,170	117,000		8,190	735,500
Average price of all commodities (Price of composite unit)				$78\frac{74}{117}$			$90\frac{70}{117}$			100			$89\frac{659}{819}$

number of the new units, $\frac{10}{9}$ bushel × 90 = 1 bushel × 100. This, of course, gives the identical total value, 9000. Likewise with sugar and wine. Sugar rated at 600 per cwt. is equivalent to a rating of 100 per $\frac{1}{6}$ cwt. And dividing the unit into six units increases the number of units sixfold, or 160 becomes 960. In case of wine, 100 substituted for 300 as the unit price changes the unit of quantity from 1 gallon to $\frac{1}{3}$ gallon, and the number of units from 40 to 120. The *price* columns and the *number-of-units* columns of the various periods sustain the same ratios between the two tables as those of the base periods.

Table III is constructed on the same plan as Table IV, the base period being the second instead of the third. The total value columns of I, III, IV are identical, as logically they should be. Table II exhibits a variation in this respect, as it is constructed on a slightly different plan, to show that different methods may be employed, if the application of principle is rigorous. In Table II, instead of changing the unit of quantity from 1 bushel to $\frac{5}{4}$ bushel for wheat, to compensate for the substitution of 100 for 80 in the price, the unit 1 bushel is retained, which necessitates the retention of 125 as the number of units. This raises the total value to 12,500. Now every individual transformation of this table must take account of this arbitrary manipulation of the price of wheat. This is done, in case of sugar, by changing the unit of quantity to $\frac{1}{5}$ cwt., instead of $\frac{1}{4}$ cwt., when 100 is substituted for 400 in the price column. This preserves the proportions between the corresponding data for wheat and sugar. 1 : $\frac{1}{5}$: : 400 : 80. That is, if 80 and 400 are each to be represented by 100, and the unit of the 80 commodity to remain unchanged, then, to preserve the original relations, the unit of the 400 commodity requires to be divided into 5 units. Of course, the number of units for sugar must be increased fivefold. A simpler way to get at this transformation is to increase the figures in the price columns of Table I 25 per cent. uniformly. Then the transformation is reduced to the method employed with III and IV. This is equivalent to changing the unit of value. If 80 is replaced by 100, that means that we have 100 *eight tenths of-a-cent* units instead of 80-*cent* units, so that the increased figures do not change the original values. The total value columns have their figures increased 25 per cent. That is, instead of 80, we have 100; instead of 400 we have 500, or $\frac{5}{4}$ instead of 1. But the unit is reduced by the same ratio. Instead of 1c. the unit is $\frac{4}{5}$c., so that the corresponding columns in the two tables express identical values. This transformation is more complicated than the other, because it involves a change in the unit of value as well as changes in the units of quantity. It is given merely as an indication of the protean possibilities of systems of index numbers.

The point of vital importance in these tables, is the absolute correspondence. Not only do the prices of the single commodities for the various periods and their averages exhibit the same proportions in all the tables, but the average prices of all the commodities for any single period, and the general averages are rigorously proportionate. Test the results of Tables I and II:

	1	2	3	General average
Averages by Table I	$3.06\frac{2}{3}$	$3.53\frac{1}{3}$	3.90	$3.50\frac{5}{21}$
Averages by Table II	100	$115\frac{5}{23}$	$127\frac{4}{23}$	$114\frac{67}{322}$

$$\frac{100}{306\frac{2}{3}} = \frac{115\frac{5}{23}}{353\frac{1}{3}} = \frac{127\frac{4}{23}}{390} = \frac{114\frac{67}{322}}{350\frac{5}{21}} = \frac{15}{46}$$

or, taking the ratios between corresponding periods, including the general average, in the two tables, we have

$$\frac{3.53\frac{1}{3}}{3.06\frac{2}{3}} = \frac{115\frac{5}{23}}{100} = \frac{53}{46}$$

$$\frac{3.90}{3.06\frac{2}{3}} = \frac{127\frac{4}{23}}{100} = \frac{117}{92}$$

$$\frac{3.50\frac{5}{21}}{3.06\frac{2}{3}} = \frac{114\frac{67}{322}}{100} = \frac{1471}{1288}$$

Similar relations exist between any two tables. The movement of price is indicated with absolute uniformity and unerring accuracy by all the tables.

In the case of disproportionate quantities there is a different story to chronicle. In order to present striking results, let us follow Professor Pierson's plan of supposing cases that exhibit

TABLE V.

Commodity	Unit	Period 1 Price	Period 1 Units (Quantity)	Period 1 Total value	Period 2 Price	Period 2 Units (Quantity)	Period 2 Total value	Period 3 Price	Period 3 Units (Quantity)	Period 3 Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
A..................	1 bu.	$ .05	× 10 =	$ .50	$ 1.00	× 73 =	$ 73.00	$ .25	× 972 =	$243.00	$ .30	× 1055 =	$ 316.50
B..................	1 cwt.	20.00	× 5 =	100.00	34.00	× 1 =	34.00	4.00	× 104 =	416.00	5.00	× 110 =	550.00
C..................	1 gal.	2.00	× 84 =	168.00	.50	× 24 =	12.00	40.00	× 7 =	280.00	4.00	× 115 =	460.00
Totals..................			99)	$268.50		98)	$119.00		1083)	$939.00		1280)	$1,326.50
Average price of all commodities (Price of composite unit)				$ $2.71\frac{21}{99}$			$ $1.21\frac{42}{98}$			$ $.86\frac{762}{1083}$			$ $1.03\frac{810}{1280}$

TABLE VI.

Commodity	Unit	Period 1 Price	Period 1 Units (Quantity)	Period 1 Total value	Period 2 Price	Period 2 Units (Quantity)	Period 2 Total value	Period 3 Price	Period 3 Units (Quantity)	Period 3 Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
A..................	20 bu.	100	× $\frac{1}{2}$ =	50	2000	× $3\frac{13}{20}$ =	7,300	500	× $48\frac{12}{20}$ =	24,300	600	× $52\frac{15}{20}$ =	31,650
B..................	$\frac{1}{20}$ cwt.	100	× 100 =	10,000	170	× 20 =	3,400	20	× 2080 =	41,600	25	× 2200 =	55,000
C..................	½ gal.	100	× 168 =	16,800	25	× 48 =	1,200	2000	× 14 =	28,000	200	× 230 =	46,000
Totals..................			$268\frac{1}{2}$)	26,850		$71\frac{13}{20}$)	11,900		$2142\frac{12}{20}$)	93,900		$2482\frac{15}{20}$)	132,650
Average price of all commodities (Price of composite unit)				100			$167\frac{3445}{7165}$			$43\frac{17682}{21426}$			$53\frac{106425}{248275}$

TABLE VII.

Commodity	Unit	Period 1 Price	Period 1 Units (Quantity)	Period 1 Total value	Period 2 Price	Period 2 Units (Quantity)	Period 2 Total value	Period 3 Price	Period 3 Units (Quantity)	Period 3 Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
A..................	1 bu.	5	× 10 =	50	100	× 73 =	7,300	25	× 972 =	24,300	30	× 1055 =	31,650
B..................	$\frac{1}{34}$ cwt.	$58\frac{28}{34}$	× 170 =	10,000	100	× 34 =	3,400	$11\frac{26}{34}$	× 3536 =	41,600	$14\frac{24}{34}$	× 3740 =	55,000
C..................	2 gal.	400	× 42 =	16,800	100	× 12 =	1,200	8000	× $3\frac{1}{2}$ =	28,000	800	× $57\frac{1}{2}$ =	46,000
Totals..................			222)	26,850		119)	11,900		$4511\frac{1}{2}$)	93,900		$4852\frac{1}{2}$)	132,650
Average price of all commodities (Price of composite unit)				$120\frac{210}{222}$			100			$20\frac{36700}{45115}$			$27\frac{16325}{48525}$

TABLE VIII.

Commodity	Unit	Period 1 Price	Period 1 Units (Quantity)	Period 1 Total value	Period 2 Price	Period 2 Units (Quantity)	Period 2 Total value	Period 3 Price	Period 3 Units (Quantity)	Period 3 Total value	Av. price for three periods	Total units (Quantity)	Total value for three periods
A..................	4 bu.	20	× $2\frac{1}{2}$ =	50	400	× $18\frac{1}{4}$ =	7,300	100	× 243 =	24,300	120	× $263\frac{3}{4}$ =	31,650
B..................	$\frac{1}{4}$ cwt.	500	× 20 =	10,000	850	× 4 =	3,400	100	× 416 =	41,600	125	× 440 =	55,000
C..................	$\frac{1}{40}$ gal.	5	× 3360 =	16,800	$1\frac{1}{4}$	× 960 =	1,200	100	× 280 =	28,000	10	× 4600 =	46,000
Totals..................			$3382\frac{1}{2}$)	26,850		$982\frac{1}{4}$)	11,900		939)	93,900		$5303\frac{3}{4}$)	132,650
Average price of all commodities (Price of composite unit)				$7\frac{31725}{33825}$			$12\frac{11300}{98225}$			100			$25\frac{5625}{530375}$

extraordinary variations. They are collected in Tables V to VIII. Table V exhibits the phenomena by means of actual prices. Tables VI, VII, VIII employ index numbers, the base period moving from 1 to 2 and then to 3, in the successive tables. These tables are constructed after the simple plan of Tables III and IV, so that the total value columns of the four tables are identical. As in the first set of tables, we have the variations of prices for the single commodities exactly corresponding in the four tables. We may note also that the average prices for each period are correctly computed, since any average substituted for all the prices averaged will not change the aggregate of the total value column in that period. But these latter averages have no further significance. There is no correspondence between the tables. Let us place the four sets in parallel rows:

	1	2	3	General average
Averages by Table V	2.71 +	1.21 +	.86 +	1.03 +
Averages by Table VI	100	167 +	43 +	53 +
Averages by Table VII	120 +	100	20 +	27 +
Averages by Table VIII	7 +	12 +	100	25 +

Comment on this exhibit is superfluous. The marked discrepancy of the ratios—the ratios corresponding to those that are absolutely uniform in the first set of tables—shows beyond question that the movement of general prices cannot be accurately drawn, when the quantities are disproportionate.

The rationale of this result is not far to seek. The average price for any period may be taken as the price of a composite unit. For instance, in Table V, period 1, the average price, \$2.71$\frac{21}{99}$ may be taken as the price of a composite unit made up of $\frac{10}{99}$, $\frac{5}{99}$, and $\frac{84}{99}$, respectively, of the conventional units of A, B, C. In period 2 of the same table, the composite unit corresponding to the average price, \$1.21$\frac{42}{98}$, has for its constituent elements $\frac{73}{98}$, $\frac{1}{98}$, $\frac{24}{98}$, respectively, of the same conventional units of A, B, C. These two composite units are strikingly diverse. The one contains, as compared with the other

$\frac{10}{99}$	against	$\frac{73}{98}$	of the	conventional	unit of	A
$\frac{5}{99}$	"	$\frac{1}{98}$	"	"	"	B
$\frac{84}{99}$	"	$\frac{24}{98}$	"	"	"	C

They are thus obviously dissimilar units, and no inference can be drawn as to the movement of price between the two periods. Similar divergence applies to all the composite units of this second set of tables. No two units are alike; hence no significant comparison between their prices can be drawn.

Let us turn now to the first set of tables. Table I, period 1, has a composite unit consisting of $\frac{125}{375}$, $\frac{200}{375}$, $\frac{50}{375}$, respectively, of 1 bushel of wheat, 1 cwt. of sugar, 1 gallon of wine. For period 2, the ratios are $\frac{475}{1425}$, $\frac{760}{1425}$, $\frac{190}{1425}$. These ratios are the same as those of period 1, for

$$\frac{125}{375} = \frac{475}{1425} = \frac{1}{3}$$
$$\frac{200}{375} = \frac{760}{1425} = \frac{8}{15}$$
$$\frac{50}{375} = \frac{190}{1425} = \frac{2}{15}$$

That is, the composite units for these two periods are identical, containing each $\frac{1}{3}$ bushel of wheat, $\frac{8}{15}$ cwt. of sugar, $\frac{2}{15}$ gallon of wine. The third period obviously has the same composite unit. Hence the average prices are prices of the same units or things, and comparisons are legitimate. The same is true of the three index tables of this set, and it must be true of all tables of proportionate quantities that are correctly constructed.

In the first set of tables, the periodic averages are of identical units; in the second set, they are of diverse units. In the former case, the composite unit, being constant through the various periods, may be regarded as the unit of a constructive single commodity, and its variations in price are amenable to the same laws with those of any single commodity. In fact, for our purpose, such a commodity, or constructive commodity, belongs to the same category as, say, lead-pencils, knives, spectacles, sewing-machines, a thousand and one articles which are composite, but have a stable association of their parts.

In the second set of tables, the composite unit is exceedingly unstable. It varies from period to period in every table, and the variations are so radical that comparison of prices is absurd.

One might as well talk of the movement of prices in the case of two periods, one of which shows an exclusive business in cattle, the other in railway ties. Nevertheless we need not go as far as Professor Pierson, even in the case of disproportionate quantities, and abandon the enterprise entirely. The hypothetical case of our tables is not typical in the sense of fairly representing the changes of any actual observation. It has served its purpose in substantiating the claim that general movements of prices cannot be accurately measured. But there can usually be an approximate measurement, the approximation becoming near or remote as proportionality among the quantities advances or recedes. If the quantities are fairly proportional, the method of Sauerbeck, Edgeworth, Marshall, and others may be employed, namely, the application of the quantities belonging to some one period, the initial, final or mean period, to all the periods uniformly. This method should usually give approximate results. Similar approximation may be secured without disturbing the actual quantities. The averages for each period may be accurately obtained. These averages, as explained above, will relate to diverse units, but these units will be approximately uniform to the same degree that the quantities are proportional, or approach proportionality.

It may be said in closing that there need be no lack of occasions to utilize the services of price tables in perfectly trustworthy operations. All enterprises that aim at ascertaining the periodic ratios of real wages of laboring or other classes are movements in the right direction. The budget method of Dr. Falkner and others, involving the consideration of constant quantities, should secure accurate results. Besides, the spirit of such an enterprise appeals to the approbation of all who are actuated by philanthropic motives.

R. S. Padan.

The University of Chicago.