The Stability of a Competitive Economy: A Survey Article

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THE STABILITY OF A COMPETITIVE ECONOMY: A SURVEY ARTICLE

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Some of the recent contributions to the problem of the stability of a competitive economy are surveyed. Emphases are laid on the uses of economic laws such as Walras' law, the homogeneity of demand functions with respect to prices, etc., in proving stability, and on the development of models of non-tâtonnement processes of adjustment in the market.

1. INTRODUCTION

THE THEORY of the general competitive equilibrium, as developed by Leon Walras [**64**], has recently been reformulated in terms of fairly advanced mathematical methods. The first problem studied extensively was concerned with the conditions under which a competitive equilibrium exists for a model of general equilibrium. Among contributions to the existence problem, Wald [**63**], Arrow and Debreu [**5**], McKenzie [**35**], Nikaido [**48**], Gale [**18**], and Negishi [**44**] may be cited. The optimality problem of a competitive equilibrium was also investigated by, e.g., Arrow [**2**], Debreu [**15, 16**], and, in greater detail, by Hurwicz [**27**]. However, it was not until the paper by Arrow and Hurwicz [**6**] was published that the stability problem of a competitive economy was investigated systematically within the framework of general equilibrium analysis.

The purpose of the present article is to survey some of the more important recent contributions to the theory of the stability of a competitive economy. After a general discussion of the nature of the stability problem (Section 2) and a short review of the earlier literature (Section 3), I shall proceed to define a model (Section 4) and present various results on the stability of the

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competitive equilibrium for this model (Sections 5–10). The main line of argument can be broadly summarized as follows.

Most of the studies of the stability of the price adjustment process in a competitive economy that have so far been developed deal with the tâtonnement process in which no actual exchange takes place until equilibrium is reached (Section 4).[2] The first systematic treatment of this problem is given in Arrow and Hurwicz [**6**] and in its sequel, Arrow, Block, and Hurwicz [**4**]. Extensions and generalizations are given in Arrow and Hurwicz [**7**, **8**, **9**], McKenzie [**36**, **37**], Morishima [**40**], Nikaido [**49**], Nikaido and Uzawa [**50**], and Uzawa [**58**, **60**]. Among several interesting results obtained, the most important is the global stability of the gross substitute case due to Arrow, Block, and Hurwicz [**4**] (Section 5).[3,4] It may be emphasized that economic laws such as Walras' law, the homogeneity of the demand function with respect to all prices, etc., played important roles in the proofs of stability. A suggestive contribution by Allais [**1**], which has been almost neglected so far, must also be noted (Section 6).

Scarf [**55**] offered examples of instability (Section 7). Therefore, the tâtonnement processes can be stable only under some restrictions, such as gross substitutability, etc. The tâtonnement process is, however, not the only way to equilibrium. Is there any other price adjustment process which is generally stable, or at least more stable than the tâtonnement process? The stability of various non-tâtonnement or barter processes (Section 4), in which trade is actually carried out according to certain rules at disequilibria, has been studied recently by Hahn [**22**, **23**], Hahn and Negishi [**24**], Negishi [**45**], and Uzawa [**59**]. It is interesting to note that these non-tâtonnement processes are generally more stable than the tâtonnement processes (Sections 8–10).

2. THE NATURE AND RELEVANCE OF THE STABILITY PROBLEM

In the first part of this section (2.1), various ways of dynamizing the general equilibrium model will be explained, and the one we are going to be concerned with in this article will be specified. The next subsection (2.2) will be devoted to a discussion of the need for stability analysis. Finally, various concepts of stability will be presented verbally in Section 2.3.

[2] For the tâtonnement process, see Walras [**64**], Patinkin [**52**, pp. 377–385] and Uzawa [**61**].

[3] The local stability of the same case is obtained independently by Hahn [**21**], Arrow and Hurwicz [**6**] and Negishi [**43**]. See also Newman [**46**, **47**], in which a systematic survey of the analysis of local stability is given.

[4] Among results, other than for the gross substitute case, the two-commodity case, the dominant diagonal case, the quasi-integrable case, the weak axiom of revealed preference case, and the "no trade" at equilibrium case, etc. are important. See Arrow and Hurwicz [**6**], Arrow, Block, and Hurwicz [**4**], McKenzie [**36**], Nikaido [**49**], Nikaido and Uzawa [**50**], and Uzawa [**58**, **60**].

2.1. The stability problem is concerned with the question of what happens to the time paths of economic variables, such as prices and outputs, which are generated from certain dynamic adjustment processes. If they converge to some equilibrium position, the relevant dynamic process is said to be stable. As Samuelson [**54**] observed, we cannot consider the stability of a system, say, a competitive economy, without analyzing its dynamic adjustment process. Different adjustment processes have different stability properties, so that a system involving one adjustment process may be stable while another system, identical in the static aspect with the former but coupled with a different adjustment process, may not.[5]

The static theory of general equilibrium, developed by Walras [**64**], Cassel [**14**], Hicks [**26**], Leontief [**31**], etc., can be dynamized in many different ways. Each dynamic system presents its own stability problem, though all can be analyzed by common mathematical tools. Most of the dynamic models so far developed may be grouped into one of the following two types: one type contains the "magnificent" dynamics of trade cycles and economic growth such as the Hicksian microscopic model of trade cycles [**26**, ch. XXIV], the growth model of von Neumann [**62**], and Leontief's dynamic system [**32**], and the other type contains the dynamics of the market clearing process such as the Walrasian tâtonnement [**64**] and the Hicksian process of adjustment to the temporary equilibrium within a "week" [**26**].

The models of trade cycles and economic growth generate time paths of outputs, capital stocks, and prices, which are of a dynamic equilibrium type, in which the supply of and demand for each commodity are assumed to be continuously equal in every market. This abstraction from the market clearing process, which may be considered as a shorter run phenomenon than the one under consideration, may be justified if the former is rapidly damped and can be supposed to have worked out its effects.[6]

On the other hand, the stability analyses of a competitive economy, cited in Section 1, and to be reviewed in this article, are concerned with the behavior of the short run market clearing adjustment process towards temporary equilibrium within a "week" in the sense of Hicks [**26**], i.e., within the unit period of consumption and production planning. There are models in which the market clearing adjustment process is extended over several Hicksian weeks (Arrow [**3**], Arrow and Hurwicz [**8**], Arrow and

[5] For the methodological aspect of the stability problem, see Samuelson [**54**, ch. XI] and Newman [**47**], to which this section owes very much.

[6] Samuelson [**54**, pp. 330–331]. It is also possible to disregard the slow long run changes and concentrate upon the shorter run process. For example, in a short run Keynesian model of income determination, it is often assumed that the stock of capital is fixed. Sargan [**53**] considered both long run and short run processes at once in a generalized Leontief model.

McManus [12], Arrow and Nerlove [13], and Enthoven and Arrow [17]). But such an extension is possible only under such special assumptions as that the behavior of individuals in the current period is not affected by that of past periods.

It has often been assumed in the model of the market clearing process —and in this article we shall assume—that prices move in accord with the excess demand (demand minus supply) in each market[7] and that the demand for and supply of commodities are functions of prices. The latter assumption is again justified if individual consumption and production plans respond rapidly relative to price changes.

In the case of constant returns to scale, however, the demand for and supply of commodities from a firm cannot be well-behaved functions of prices. Since in this case profit is proportional to scale at any given set of prices, the profit-maximizing scale may be infinite, if positive profits are possible at some level. Or it may be that there are zero profits at all scales, in which case profit maximization does not define the behavior of firms. Finally, if profits are negative at all positive scales, the optimal scale is zero. This is why Walras [64], while assuming instantaneous utility maximization by the consumer, did not prescribe instantaneous profit maximization for firms. He assumed that the price change has the same sign as the excess demand, and that the change in the scale of production has the same sign as the marginal profitability of scale. This case, with lagged adjustments of producers, was recently treated by Morishima [40]. The excess demand model with lagged consumers' adjustments was, on the other hand, treated by Arrow and Hurwicz [6].

The Marshallian process in which the response of output is governed by the excess demand price (demand price minus supply price) is not a market clearing process within a "week" but a long run process requiring several weeks in the sense of Hicks. Hicks [26, p. 62] argued, moreover, that the Marshallian process is appropriate to monopoly rather than to competition.

2.2. Let us now consider the reason for stability analysis. The existence of general equilibrium is rigorously proved in the literature cited in Section 1 above. Does the existence proof not assure that equilibrium is really established? Here are the reasons why, in addition to the existence proof of equilibrium, some representation of the adjustment outside equilibrium must be provided, and its stability required, if the model of a competitive economy is to be entertained as a good description of the facts.

As Walras [64] observed, the equilibrium we obtain mathematically or

[7] Koopmans ([28 p. 179]) is critical of this model on the ground that it is difficult to identify this assumption about market behavior with any individual behavior. See Section 4 below.

theoretically is established empirically or practically in the market by the mechanism of competition. At the beginning of every period, markets are not necessarily in equilibrium, i.e., the supply of and demand for commodities are not necessarily equal, and the market clearing adjustment process begins to work. The competition of buyers and sellers alters prices. Prices rise for those commodities whose demand exceeds supply, and fall for those commodities where the reverse holds. We know from experience that under this process prices usually do not explode to infinity or contract to zero, but converge to an equilibrium such that the supply of and demand for commodities are equal. Hence, the process which we choose to represent reality must display the same stability. We must therefore search for intuitively appealing and widely acceptable conditions or restrictions on the model that are sufficient to ensure stability.

The equilibrium once established in this way is continuously subject to changes and disturbances, such as of taste, technology, resources, and weather. Suppose the system, which has been in equilibrium, is thrown out of it by some of those changes or disturbances. It is known empirically that the economy is in fact fairly shock-proof. Dynamic market forces are generated which bring the economy back to equilibrium when it is perturbed, i.e., there exists a stable adjustment process when the economy is out of equilibrium. Realistic economic models should contain such a dynamic equilibrating process.

Furthermore, theories of trade cycles and of economic growth that are of the dynamic equilibrium type cited in Section 2.1 assume that temporary or short run equilibrium is easily and quickly established in each period and rapidly recovered when disturbed by shocks. Studies of the stability of the market clearing process do, therefore, offer to these theories some assurance concerning their fundamental assumptions.

Since welfare economics assures us that under certain assumptions a competitive equilibrium can be identified with an economic optimum (the optimality problem cited in Section 1), we may conclude that the competitive process towards market equilibrium is also a computational device for solving the problem of optimal resource allocation (Arrow and Hurwicz [**10**], Marschak [**34**]). Indeed, Pareto [**51**] compared the market to a computing machine. Of course, the system of simultaneous equations describing the general equilibrium can be solved by some centralized procedure involving the use of computing machines rather than by the market which solves the problem under decentralization. A completely centralized organization would, however, require a capacity for the storage and processing of technological and other information that exceeds anything likely to be available. This is the reason why Lange [**29**] concluded that accounting prices in a socialist economy should be determined by a decentralized trial and error process in which the Central Planning Board performs the functions of the competitive market. Stability, then, is necessary for the competitive market mechanism to be a satisfactory practical device for solving the problem of optimum resource allocation.[8]

So far we have been concerned with the reasons why models should satisfy sufficient conditions for stability—conditions which ensure the stability of the dynamic process. Most of the results so far obtained in the analysis of stability, which we are going to review in this article, are also concerned with sufficient conditions. One would, however, also like to know necessary conditions. Hicks [**26**, p. 62] argued that if stability is taken for granted as an assumption of regularity, one can deduce, from the necessary condition for stability, rules of comparative statics regarding the way in which the price system will react to changes in taste or resources—called the Correspondence Principle by Samuelson [**54**]. Newman [**47**] correctly argued that such a use of necessary conditions might be illegimate, since we can not assume a priori that our model is stable just because the world is stable. It cannot be denied, however, that laws of comparative statics and of stability have a good deal to do with each other. Recently, Morishima [**41**] demonstrated this again.

2.3. In this article we shall be concerned mainly with global stability, or stability in the large, which means that economic variables generated from the dynamic process approach some equilibrium in the limit as time becomes infinite, regardless of their initial values. Samuelson [**54**] called this perfect stability of the first kind. But there are a few other concepts of stability, some of which will be referred to in this article.

If equilibrium points are not distinct from each other but cover a whole line or region, we shall be concerned with quasi-stability which implies that variables converge to the set of equilibrium points, or more technically, that every limit point of the process is an equilibrium (Uzawa [**60**]). To prove quasi-stability, it is sufficient to show that there is some continuous function of the relevant variables which is decreasing through time at disequilibria and that the domain of the variables is bounded, i.e., they do not go to positive or negative infinity. It is intuitively clear that stability is obtained if the distance in any sense from the present position to an equilibrium, or to a set of equilibria, is decreasing through time.[9]

The decrease of a continuous function of the variables, which is used in

[8] Also, convergence of the process to some neighborhood of the equilibrium must be rapid enough for practical purposes.

[9] Distance from point x to point y is a nonnegative real valued continuous function $D(x, y)$ such that $D(x, y) = 0$ if and only if $x = y$, $D(x, y) = D(y, x)$, and $D(x, z) + D(z, y) \geqslant D(x, y)$ for any third point z. Any function which satisfies these conditions

the proof of quasi-stability, is a sort of mathematical extension of the concept of decreasing distance.[10] Global stability follows from quasi-stability when equilibrium is unique, or when equilibria are distinct from each other.

If the dynamic process is converging only when initial values of the variables are close to equilibrium values, we speak of local stability or stability of the first kind in the small. To prove stability in the small of a dynamic process, it is sufficient to consider a linear system approximating the process in the neighborhood of an equilibrium point. If equilibrium is unique, global stability implies local stability, but not vice versa. Sometimes global stability may seem to be rather too stringent a condition to impose on the system and, as Newman [**47**] states, not always to be preferred to local stability, since in every period the market clearing process starts from the historically given values of variables which are close to equilibrium rather than from "prix criés au hasard," and most of the disturbances or shocks in the economy are likely to be small in fact. Local stability, however, is sometimes unsatisfactory, since it is quite possible for there to be multiple equilibria, none of which is completely stable from the local point of view (attracting all neighboring points), while the system is, in its entirety, globally stable (approaches some equilibrium). In the study of global stability, we are concerned with the behavior of a whole system, say, a competitive economy, rather than with the stability of a particular equilibrium.

There are several other concepts of stability which we shall not be concerned with in this article. Among them, perhaps the most important is stability in the sense of Lyapunov [**33**]. This implies that variables remain close to equilibrium, without necessarily converging to it, when perturbed slightly.[11] Samuelson's stability of the second kind [**54**] corresponds to this. In the theory of trade cycles, we have another concept of stability, that of orbital stability which means that the time paths of the variables converge not to a point but to the path of some periodic motion, i.e., in the long run the same cycle is repeated (Goodwin [**19**]). In growth theory, relative stability, i.e., the convergence of the ratio of variables to that of the balanced growth path, is important (Solow and Samuelson [**56**]).

3. HISTORICAL REMARKS

Before dealing with recent contributions, it may not be out of place to

is called a distance, though the most natural concept is provided by Euclidean distance. The distance of x from the given equilibrium $\bar{x}$ can be considered a continuous function of x.

[10] This method of proof of stability is originally due to Lyapunov [**33**]. See also Hahn [**25**].

[11] Scarf's example [**55**] of instability is in fact stable in the sense of Lyapunov, though not in the sense of our definition of stability.

give a brief historical account on some of the early work on the stability analysis of a competitive economy.

Walras [**64**] offered two methods for solving the equations of the general economic equilibrium, the theoretical or mathematical solution, and the empirical or practical solution of the market. The former was merely to count the number of unknowns and of equations. This was criticized, among others, by Wald [**63**]. Recent studies on the existence of an equilibrium, cited in Section 1, solved this problem completely. The latter solution is that of tâtonnement, a trial and error process representing the market mechanism under free competition.[12] The stability of tâtonnement, in which prices change in accord with excess demands, was not successfully shown by Walras, except for the case of the exchange of two commodities. The Walrasian stability condition for the case of two commodities is that if the price of one commodity in terms of the other (i.e., numéraire) is above the equilibrium price, there is an excess supply for that commodity, and if below equilibrium, an excess demand. Since the price rises if there is excess demand and falls if there is excess supply in the tâtonnement, this stability condition implies that there are forces to bring the price back to the equilibrium.

The stability condition, given by Walras in the two commodity case, was generalized by Hicks [**26**] for the many commodity case. In order for equilibrium to be perfectly stable, according to Hicks, a rise of the price of any commodity above the equilibrium must be accompanied by an excess supply of that commodity, and a fall below the equilibrium by an excess demand, so that a force is generated to bring the changed price back to equilibrium. This behavior must hold regardless of the state of other markets, i.e., whether or not other prices are unchanged or adjusted so as to maintain equilibrium in the relevant markets. More technically, the sign of the derivative of excess demand of a commodity with respect to its own price must be negative, even when any arbitrary subset of other prices are kept unchanged while the remaining ones are adjusted so as to maintain equilibrium in the respective markets. This implies that the sign of the principal minors of the matrix,

$$A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots \\ \vdots & \cdots & a_{ij} \cdots & \vdots \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

[12] Goodwin [**20**] insisted that Walrasian tâtonnement is merely a mathematical device to solve the equilibrium problem and not a representation of the adjustment process of competitive markets, an interpretation with which we can not agree. See Patinkin [**52**, pp. 377–385].

be alternatively negative and positive.[13] Here a_{ij} is the partial derivative of the excess demand of the ith commodity with respect to the price of the jth commodity evaluated at equilibrium. When a matrix satisfies this condition, it is often called Hicksian. The Hicksian stability condition, though useful in comparative statics, remained static in nature since it was obtained without fully exploring the dynamics of the market adjustment process. Contributions by Mosak [**42**] and Sono [**57**] were also of the same static variety.

It is Samuelson [**54**] who, criticizing Hicks, first observed that we cannot consider the stability problem without specifying a dynamic adjustment process. He formulated the problem as a set of dynamic equations and gave mathematical conditions for the convergence of its solution, i.e., the stability of the equilibrium. For example, he considered the stability of a differential process, in which the instantaneous rate of change of the price of any good is proportional to its excess demand, the latter being regarded as a function of all prices. If excess demands are approximated linearly at equilibrium and speeds of adjustment (ratios of the rate of change of price to excess demand) are set equal to one, the stability condition is that the real part of all the characteristic roots of the matrix A above should be negative. This true dynamic stability condition is generally different from the Hicksian condition. The latter is neither necessary nor sufficient for the former.

Since the Hicksian condition is useful in comparative statics, the relationship between dynamic stability and Hicksian stability was explored by, e.g., Samuelson [**54**], Lange [**30**], Metzler [**38**] and Morishima [**39**], with the following results.

(i) If the matrix A is symmetrical, i.e., $a_{ij} = a_{ji}$, the Hicksian condition and the true dynamic condition coincide with each other (Samuelson, Lange).

(ii) If the matrix A is quasi-negative-definite,[14] both Hicksian and dynamic stability conditions are satisfied (Samuelson).

(iii) The Hicksian condition is necessary if the dynamic process is stable regardless of the values of speeds of adjustment (Metzler).

(iv) If the matrix A has all off-diagonal elements positive ($a_{ij} > 0$, $i \neq j$), i.e., all goods are gross substitutes, the Hicksian and the dynamic conditions coincide (Metzler).

Samuelson and his followers did not, however, take full advantage of the implication of the assumptions underlying the perfectly competitive model.[15]

[13] In the case of two commodities, the Hicksian condition coincides with the Walrasian condition.

[14] A matrix A is quasi-negative-definite if $[A + A']/2$ is negative-definite, where the prime implies transposition. See Samuelson [**54**].

[15] See Arrow and Hurwicz [**6**]. Perhaps Allais [**1**] who worked independently of Samuelson and others may be an exception in this respect.

Also, in most cases they examined stability in a small neighborhood of the equilibrium, i.e., local stability, by the method of linear approximation.

The nature of the competitive economy in its relation to the stability of the price adjustment process was first fully explored by Hahn [**21**], Arrow and Hurwicz [**6**], and Negishi [**43**]. It was proved, by use of Walras's law (Hahn, Arrow, and Hurwicz) or the homogeneity of demand functions with respect to all prices (Negishi), that if all goods are gross substitutes, i.e., $a_{ij} > 0$, for all $i \neq j$ in the matrix A, not only do Hicksian and dynamic conditions coincide, as stated above, but also that dynamic stability itself necessarily holds. Our understanding of stability in the large, i.e., global stability, with due attention to the nonnegativity of prices,[16] was developed by Arrow, Block, and Hurwicz [**4**][17] and many others, such as McKenzie [**37**], Nikaido [**49**], Nikaido and Uzawa [**50**], etc.

4. CONSTRUCTION OF THE MODEL

4.1. We are now going to concentrate attention on a particular model in order to make our discussion more precise. In this subsection let us construct a static model of a pure exchange economy and derive the aggregate demands for commodities as functions of all prices. A model of the dynamic processes will be given in the next subsection. We confine ourselves to the case of a pure exchange economy for two reasons. First, almost all the essential problems in the stability analysis occur even in this simplest model of the economy; and, secondly, many of the works we are going to survey are studies of this case. The results in Sections 5 and 6 can, however, be extended to an economy with production, since the assumptions utilized in those sections concern aggregate excess demands which are derived as functions of all prices, when production plans react rapidly and well-behavedly, i.e., continuously to changes of prices (Section 2.1). Results in Sections 7–10, on the other hand, depend essentially on the properties of a pure exchange economy and no attempt has been made so far to extend them to models with production.

Suppose certain amounts of the initial stocks of commodities are distributed to each individual participant in the economy. Since there is no production, total stocks of commodities in the economy remain unchanged. Each individual strives to maximize his utility through exchanges of commodities subject to the price ratios which are given in the market. As a result, for each commodity a quantity is demanded which is a function of all

[16] The nonnegativity problem in the study of the gradient method (a method to solve the programming problem) is developed in Arrow, Hurwicz, and Uzawa [**11**].

[17] The method used to prove stability in Arrow, Block, and Hurwicz [**4**] is generalized in Uzawa [**60**] from the point of view of Lyapunov [**33**].

prices. If this quantity happens to be equal to the total existing amount in the economy for each commodity, market prices are called equilibrium prices relative to the initial distribution of commodities. It must be noted that different initial distributions generally generate different equilibrium prices. Although this model is very simple, two fundamental economic laws can be derived: Walras' law and the homogeneity of demand functions with respect to prices. The former implies that the value of total demand of all commodities is always equal to that of total supply, while the latter implies that demand is not affected by a proportionate change of all prices.

Let there be n individual participants labelled $i = 1, \ldots, n$ and m commodities labelled $j = 1, \ldots, m$ in the economy. Let us use the following notation:

P_j is the price of the jth commodity;
$\bar{X}_{ij}$ is the holding of the jth commodity by the ith individual, assumed to be nonnegative;
$\bar{X}_j$ is the total amount of the jth commodity, a constant, $\Sigma_i \bar{X}_{ij} = \bar{X}_j$;
X_{ij} is the demand for the jth commodity by the ith individual, assumed to be nonnegative;
X_j is the total demand for the jth commodity, $\Sigma_i X_{ij} = X_j$;
I_i is the income (wealth) of the ith individual, $I_i = \Sigma_j P_j \bar{X}_{ij}$; and
U_i is the utility of the ith individual, a function of $X_{i1}, \ldots, X_{im}$.[18]

It is assumed that the demand for the jth commodity by the ith individual $X_{ij}(P_1, \ldots, P_m, I_i)$ is the unique solution obtained by maximizing the utility function $U_i(X_{i1}, \ldots, X_{im})$ subject to the budget constraint $\Sigma_j P_j X_{ij} = I_i$.

An equilibrium price vector $P = (P_1, \ldots, P_m)$ for a given distribution matrix of the stock of commodities among individual participants

$$\bar{X} = \begin{bmatrix} \bar{X}_{11} & \cdots & \bar{X}_{1m} \\ \vdots & & \vdots \\ \cdots & \bar{X}_{ij} & \cdots \\ \vdots & & \vdots \\ \bar{X}_{n1} & \cdots & \bar{X}_{nm} \end{bmatrix}$$

is defined as a positive price vector $P = (P_1, \ldots, P_m)$ which satisfies the condition of equality of demand and supply for each commodity,

$$\text{(e)} \qquad X_j(P, \bar{X}) = \bar{X}_j, \qquad \text{for all } j.$$

Such an equilibrium is known to exist under certain conditions which may include the positiveness of $\bar{X}$.

[18] We shall also admit the assumptions which are usually imposed on utility functions, such as differentiability, non-saturation (i.e., for any x, there is some y such that $U(y) > U(x)$), and strict quasi-concavity (i.e., for any x, the set of y such that $U(y) \geqslant U(x)$ is strictly convex). See Arrow and Debreu [**5**], Hicks [**26**].

From the definition of X_{ij}, it follows that X_j is positively homogeneous of degree zero in P, so that

$$\text{(h)} \qquad X_j(\lambda P,\bar{X}) = X_j\,(P,\bar{X}), \qquad \text{for any } \lambda > 0,$$

and if $\bar{P}$ is an equilibrium, $\lambda\bar{P}$ is also an equilibrium for any $\lambda > 0$.

Summing up the individual budget constraints, we have Walras's law,

$$\text{(W)} \qquad \Sigma_j P_j X_j = \Sigma_j P_j \bar{X}_j\,.$$ [19]

4.2. A general economic equilibrium was represented above by a set of conditions (e). Such an equilibrium is established in the market through competition among individual participants. We may introduce various models of dynamic processes to represent this phenomenon. In this article, we are concerned with two differential equation models, to be called respectively the tâtonnement and the non-tâtonnement processes.

A tâtonnement process of price adjustment is governed by the differential equation system,[20]

$$\text{(T)} \qquad \frac{dP_j}{dt} = X_j(P,\bar{X}) - \bar{X}_j \qquad (j = 1,\ldots, m)$$

where t denotes time and $\bar{X}$ remains constant through time. This is a simplified version of a modern formulation of the Walrasian tâtonnement (Samuelson [**54**, p. 270])[21] which represents the well known "law of supply and demand": the price of a commodity rises if demand exceeds supply and

[19] Although in the following we state (h) and (W) as independent assumptions, it must be noted that they are both derived from the assumption of utility maximization of individual participants.

[20] This system is called the non-normalized system. A normalized system with numéraire is

$$\frac{dP_j}{dt} = X_j - \bar{X}_j \qquad (j = 1,\ldots, m-1),$$

$$P_m = 1\,.$$

For the normalization and its relation to stability, see Arrow, Block, and Hurwicz [**4**]. A system

$$\frac{dP_j}{dt} = a_j(X_j - \bar{X}_j), \qquad a_j > 0 \text{ (constant)}$$

can be reduced to (T) by a suitable choice of units of measurement of commodities. See Arrow and Hurwicz [**6**].

[21] A formulation by a difference equation system, which is close to the original Walrasian version of tâtonnement, is given in Uzawa [**61**]. See also Allais [**1**]. In the original version of tâtonnement due to Walras [**64**], it is assumed that the adjustment takes place not simultaneously in all markets but successively in one market after another.

falls in the opposite case. In the case of the ideally well organized market, such as the stock exchange, grain markets, and fish markets, we may imagine for each commodity an auctioneer who, as an incarnation of the competitive force in the market, raises the price of the commodity at a rate proportional to the difference between demand and supply. Each individual regards the market price announced by the auctioneer as a given datum to which he must adjust himself, although the announced price is the result of the decisions of all individuals in the market. Each individual then reports his decision on demand to the auctioneer. In the case of a less organized market, we must admit that it is a serious question as to whose behavior is expressed by (T). The existing literature is quite ambiguous in this respect (see the discussion by Koopmans [**28**, p. 179]).

In this provisional process, recontract is always possible and no actual trade of commodities among individual participants is permitted (Walras [**64**] suggests the use of tickets), until the equilibrium is reached, i.e., until the process itself is terminated. Since there are no exchange transactions in the process, the distribution of commodities $\bar{X}$ remains constant over time, and we may omit it from the demand function.

A solution of this process through the initial price vector P^o is an m-dimensional function $P(t; P^o)$ of time such that

$$P(0\ ; P^o) = P^o$$

and the jth component P_j of P satisfies the identity

$$\frac{dP_j}{dt} = X_j[P(t\ ; P^o)] - \bar{X}_j \qquad \text{for } t \geqslant 0\,.$$

On the other hand, a non-tâtonnement process is governed by the differential equation system,

$$\text{(NT)} \qquad \begin{aligned} \frac{dP_j}{dt} &= X_j(P, \bar{X}) - \bar{X}_j && (j = 1, \ldots, m), \\ \frac{d\bar{X}_{ij}}{dt} &= F_{ij}(P, \bar{X}) && (i = 1, \ldots, n;\ j = 1, \ldots, m). \end{aligned}$$

In this process, the distribution of commodities $\bar{X}$ is no longer constant since some trade out of equilibrium is permitted according to certain transaction rules which are incorporated in the form of the functions F_{ij}. Functions F_{ij} denote that opportunities for individuals to change their stocks of commodities $\bar{X}_{ij}$ by exchanges with other individuals are assumed to depend on prices and on the distribution of commodities in the economy.

Because we are discussing the case of pure trade, in which total amounts of commodities, the $\bar{X}_j$'s are constant, we must impose on the functions F_{ij} the conditions

(C) $$\Sigma_i F_{ij}(P, \bar{X}) = 0 \qquad (j = 1, \ldots, m),$$

so that we have $\Sigma \dot{\bar{X}}_{ij} = 0$.[22]

A solution of this process through the initial price vector P^o and the initial distribution of commodities $\bar{X}^o$ is an $m(n+1)$ dimensional function $[P(t; P^o, \bar{X}^o), \bar{X}(t; P^o, \bar{X}^o)]$ of time such that

$$P(0\ ; P^o, \bar{X}^o) = P^o,$$
$$\bar{X}(0\ ; P^o, \bar{X}^o) = \bar{X}^o,$$

and

$$\frac{dP_j}{dt} = X_j[P(t\ ; P^o, \bar{X}^o), \bar{X}(t\ ; P^o, \bar{X}^o)] - \bar{X}_j\ ,$$
$$\frac{d\bar{X}_{ij}}{dt} = F_{ij}[P(t\ ; P^o, \bar{X}^o), \bar{X}(t\ ; P^o, \bar{X}^o)]$$

for $t \geqslant 0$ and all i, j.

In a non-tâtonnement process (NT), not only prices but also the distribution of commodities are adjusted so as to satisfy the condition of the equilibrium (e). On account of the redistribution of incomes among individual participants due to changes of prices in the midst of trading (Hicks [**26**, pp. 127–129]), the competitive equilibrium reached by a non-tâtonnement process is generally different from the one reached by a tâtonnement process. In the case of non-tâtonnement (NT), a price vector P and a distribution matrix $\bar{X}$ is called an equilibrium if

(e′) $$X_j\,(P, \bar{X}) = \bar{X}_j \qquad \text{for all } j.$$

It must be noted that, on account of (W), either in (T) or in (NT) the solution $P(t)$ remains bounded. Differentiating $\Sigma_j P_j^2(t)$, we have from (W),

$$\frac{d(\Sigma_j P_j^2(t))}{dt} = 2\,\Sigma_j P_j(t)\,\dot{P}_j(t) = 2\,\Sigma_j P_j(X_j - \bar{X}_j) = 0$$

and, therefore,

$$\Sigma_j P_j^2(t) = \Sigma_j P_j^2(0) \qquad \text{for any } t.$$

On the other hand, in (NT), $\bar{X}(t)$ remains bounded since we have from (C),

$$\Sigma_i \bar{X}_{ij}(t) = \bar{X}_j \qquad \text{for any } t.$$

A differential system is said to be globally stable if, for any given initial values, the solution of the system through those values converges to some equilibrium of the system. In the case of (T), this means $\lim_{t\to\infty} P(t) = P^*$ where $X_j(P^*) = \bar{X}_j$ for all j; and, in the case of (NT), $\lim_{t\to\infty} P(t) = P^*$, together with $\lim_{t\to\infty}\bar{X}(t) = \bar{X}^*$ where $X_j\,(P^*, \bar{X}^*) = \bar{X}_j$ for all j. A system

[22] A dot on the variable signifies d/dt.

is said to be quasi-stable if, for any initial values, every limit point of the solution of the system is an equilibrium: in the case of (T), $\lim_{v\to\infty} P(t_v) = P^*$ where $X_j(P^*) = \bar{X}_j$ for all j, and $t_v \to \infty$ as $v \to \infty$; in the case of (NT), $\lim_{v\to\infty} P(t_v) = P^*$, $\lim_{v\to\infty} \bar{X}(t_v) = \bar{X}^*$ where $X_j(P^*, \bar{X}^*) = \bar{X}_j$ for all j, and $t_v \to \infty$ as $v \to \infty$. Since the relevant variables $P(t)$ and $\bar{X}(t)$ are bounded, it is sufficient for quasi-stability that there exists a continuous function $V\{P(t)\}$ (or $V\{P(t), \bar{X}(t)\}$) which is strictly decreasing through time unless $X_j = \bar{X}_j$ for all j. If equilibrium is unique, or if the equilibria are distinct from each other, quasi-stability coincides with global stability (Uzawa [**60**]).

In the following three sections we discuss the stability and the instability of the tâtonnement process, and in the last three sections we treat the non-tâtonnement process.

5. GROSS SUBSTITUTABILITY (ARROW, BLOCK, AND HURWICZ)

This section is devoted to one of the most important results obtained in studies of the stability of the tâtonnement process (T), i.e., the global stability under gross substitutability due to Arrow, Block, and Hurwicz [**4**].[23]

5.1. The basic idea of the gross substitutability concept (Metzler [**38**], Mosak [**42**]) is this: If the price of one commodity goes up while all other prices remain unchanged, there will be an increase in demand for every commodity whose price has remained constant. Mathematically,

$$\text{(S)} \qquad \frac{\partial X_j}{\partial P_k} > 0, \qquad \text{for all } P;\, j \neq k.$$

It must be noted that (S) implies $\partial X_j/\partial P_j < 0$, since from ($h$) we have $\sum_k (\partial X_j/\partial P_k)\, P_k = 0$ for all j. Therefore, in the case of two commodities, gross substitutability implies the Walrasian condition for stability (Section 3). The reader should be careful not to confuse gross substitutability with (net) substitutability, i.e., the positive effect of a change in the price of commodity k on the demand for commodity j when real income is properly compensated (Hicks [**26**]).

An example of gross substitutability may be found in Arrow and Hurwicz [**6**, p. 550]. Suppose the utility function of each individual is of the form

$$U_i(X_{i1}, \ldots, X_{im}) = \textstyle\sum_j \alpha_{ij} \log X_{ij} \qquad \text{for all } i\,,$$

where α_{ij}'s are constants such that $\alpha_{ij} > 0$, for all i, j, and $\sum_j \alpha_{ij} = 1$. Then

[23] Extensions of the stability theorems for the case of weak gross substitutability ($\partial X_j/\partial P_k \geqslant 0$ for $j \neq k$) are given in Arrow and Hurwicz [**8**, **9**], McKenzie [**37**], and Uzawa [**60**].

we have

$$X_{ij} = \alpha_{ij} \frac{I_i}{P_j}$$

where $I_i = \sum_j P_j \bar{X}_{ij}$, and from this we obtain

$$\frac{\partial X_{ij}}{\partial P_k} > 0 \qquad \text{for all } P > 0, \bar{X} > 0, j \neq k\,.$$

Gross substitutability (S) is obtained by summing over i.

The balance of this subsection is devoted to a lemma due to Arrow, Block, and Hurwicz [**4**] concerning the uniqueness of the equilibrium price vector. Of course, the equilibrium price vector cannot be unique in the strict sense of the word. By (h), if $\bar{P}$ is an equilibrium, $\lambda\bar{P}$ for any $\lambda > 0$ is also an equilibrium. It will, however, be shown, under gross substitutability, that the equilibrium price ratios, or the equilibrium price ray, are unique, i.e., if $\bar{P}$ is an equilibrium, every other equilibrium can be expressed as $\lambda\bar{P}$ for some $\lambda > 0$. This uniqueness of the equilibrium ray is important for the proof of stability.

Now we state:

LEMMA 1: *Under assumptions of positive homogeneity (h) and gross substitutability (S) of the demand function, the equilibrium price vector $\bar{P}$ is determined uniquely up to a scalar multiple.*

A sketch of the proof is given below (Arrow, Block, and Hurwicz [**4**, Lemma 4]).

Suppose both $\bar{P}$, $\bar{\bar{P}}$ ($\bar{P} \neq \lambda\bar{\bar{P}}$ for any $\lambda > 0$) satisfy (e), and $\bar{P}_J/\bar{\bar{P}}_J = \min_j (\bar{P}_j/_j\bar{\bar{P}})$. By (h) we may replace $\bar{\bar{P}}$ by $\hat{P}$ such that $\hat{P} = \mu\bar{\bar{P}}$, $\mu > 0$ and $\bar{P}_J = \hat{P}_J$. Then $\hat{P} \leqslant \bar{P}$ and hence by, (S) and (e), $X_J(\hat{P}) < X_J(\bar{P}) = \bar{X}_J$, which contradicts the assumption that $\bar{\bar{P}}$ is an equilibrium (Wald [**63**]).

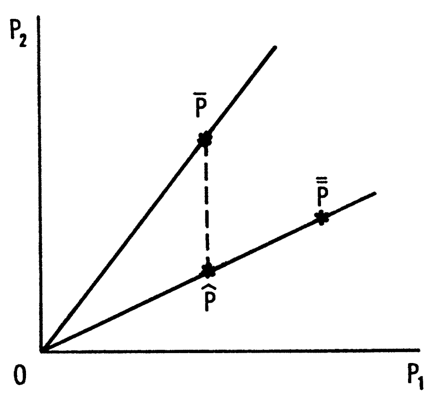

FIGURE 1

In the case of two commodities, $m = 2$, the argument above can be seen graphically (see Figure 1). Here we have $J = 1$, since $\bar{P}_1/\bar{\bar{P}}_1 < \bar{P}_2/\bar{\bar{P}}_2$. From (h) we have $X_1(\bar{\bar{P}}) = X_1(\hat{P})$. Comparing $\hat{P}$ with $\bar{P}$, we have, from (S), $X_1(\hat{P}) < X_1(\bar{P})$, since $\bar{P}_2$ is higher than $\hat{P}_2$, while $\bar{P}_1 = \hat{P}_1$. Therefore, $X_1(\bar{\bar{P}}) \neq \bar{X}_1$ and $\bar{\bar{P}}$ does not satisfy (e). Condition (e) is satisfied only by price vectors which are expressed as $P = \lambda\bar{P}$ for some positive scalar λ.

From this lemma, we know that, under assumption (S), there exists a unique equilibrium ray such as $0\bar{P}$ in the figure. Since in (T) we have $\sum_j P_j^2(t) = \sum_j P_j^2(0)$ as is stated in Section 4.2, the equilibrium we can reach, if possible, from the given initial value $P(0)$ is also unique.

If gross substitutability (S) is assumed, we can show the stability of the tâtonnement process in various ways. Since equilibrium is unique for given initial conditions, quasi-stability implies global stability. To prove the former, it is sufficient to find a continuous function which is decreasing through time at disequilibria. Different choices of such a function offer different proofs of the same theorem, the stability of the gross substitute case. The first method of proof due to Arrow, Block, and Hurwicz ([**4**, Theorem 2]) will be given in Section 5.2. The Euclidean distance or its square in the price space (the sum of squares of the difference between prices and equilibrium prices) will serve there as the function which decreases through time. The second, intuitively more clear, method of proof, also due to Arrow, Block, and Hurwicz ([**4**, Theorem 1]) will be given in Section 5.3. Distance in terms of the maximum norm in the price space (the maximum of the differences between prices and their equilibrium values) will be shown to be decreasing through time in this proof. Finally, a proof once suggested by Allais [**1**] and developed later independently by McKenzie [**37**] that the sum of the absolute values of the excess demands multiplied by prices decreases through time will be discussed in Section 6. Although these are alternative proofs for the same result, all of them are worth reporting since different information on the behavior of prices in the market is given by each of them.

5.2. In this subsection, following a lemma concerning the relation between excess demands and prices under gross substitutability (S), the first proof of the stability of tâtonnement (T) under gross substitutability is given.

By the use of both homogeneity (h) and Walras' law (W), an important lemma is obtained under the assumption of gross substitutability (S).

LEMMA 2: *Under gross substitutability* (*S*), *homogeneity* (*h*), *and Walras' law* (*W*), *we have*

$$\sum_j \bar{P}_j \{X_j(P) - \bar{X}_j\} > 0 \,,$$

for any $P > 0$, $P \neq \lambda\bar{P}$ *for any* $\lambda > 0$.

This lemma says that the sum of excess demands at any disequilibrium price situation weighted by equilibrium prices is always positive. In other words the weak axiom of revealed preference (Samuelson [**54**]) is satisfied between the equilibrium point and any disequilibrium point.

In the case of two commodities, $m = 2$, we can show this graphically (see Figure 2).

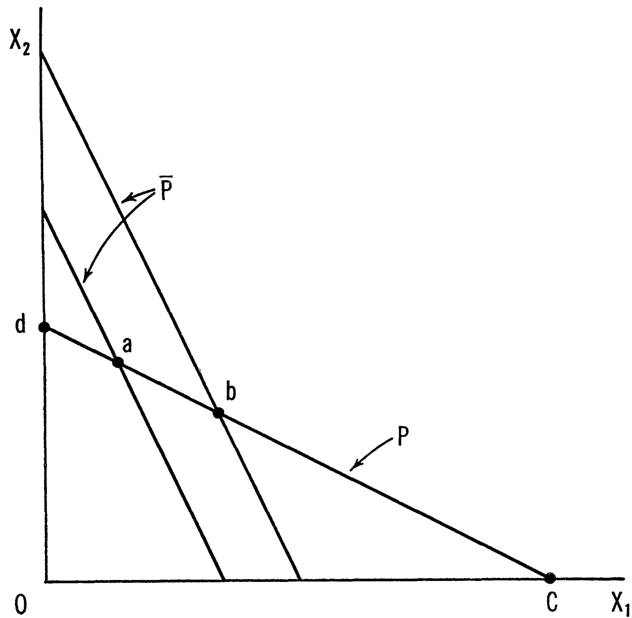

FIGURE 2

$(a = \{X_1(\bar{P}), X_2(\bar{P})\} = (\bar{X}_1, \bar{X}_2); b = \{X_1(P), X_2(P)\})$

The point a represents the total stocks of the two commodities and, by definition, total demands at equilibrium prices $\bar{P}$. The point b, corresponding to demands at P, is on the line cd which represents the price ratio of P and passes through a, since we have from (W),

$$P_1X_1(P) + P_2X_2(P) = P_1\bar{X}_1 + P_2\bar{X}_2 .$$

If $\bar{P}$ is represented by steeper lines than cd as in the figure, b must be on the line segment ac from (S). Then, comparing points a and b, we have

$$\bar{P}_1X_1(P) + \bar{P}_2X_2(P) > \bar{P}_1\bar{X}_1 + \bar{P}_2\bar{X}_2 ,$$

and the lemma holds.

By the use of this lemma, the stability of the tâtonnement process (T) is proved.[24]

THEOREM 1: *Under the assumption of gross substitutability* (S), *homogeneity* (*h*) *and Walras' law* (*W*), *the tâtonnement process* (*T*) *is globally stable.*

A sketch of the proof is as follows (Arrow, Block, and Hurwicz [**4**]):

Consider the square of the distance from the variable point $P(t)$ of the solution (T) to an equilibrium $\bar{P}$ in terms of the Euclidean norm as

$$D_2(t) = \Sigma_j (P_j(t) - \bar{P}_j)^2 ,$$

where $\bar{P}$ is normalized as

$$\Sigma_j P_j^2(0) = \Sigma_j \bar{P}_j^2 ,$$

since we know the equilibrium $\bar{P}$ is determined uniquely only up to a scalar multiple (Lemma 1) and

$$\Sigma_j P_j^2(t) = \Sigma_j P_j^2(0)$$

for all $t > 0$.

The convergence of $P(t)$ to the equilibrium $\bar{P}$ is shown by differentiating $D_2(t)$, using Walras' law (W) and Lemma 2; thus

$$\begin{aligned} \dot{D}_2(t) &= 2\Sigma_j \dot{P}_j(P_j - \bar{P}_j) = 2\Sigma_j (P_j - \bar{P}_j)(X_j(P) - \bar{X}_j) \\ &= -2\Sigma_j \bar{P}_j(X_j(P) - \bar{X}_j) < 0, \qquad \text{for } P \neq \bar{P} , \end{aligned}$$

which implies that $D_2(t)$ is decreasing through time at disequilibria.

It must be noted in the above argument that both homogeneity (h) and Walras' law (W) play essential roles.

5.3. After establishing another lemma on the excess demands under gross substitutability (S), we shall give below the second proof of the stability of the tâtonnement (T).

The following lemma is obtained by the assumptions of gross substitutability (S) and homogeneity (h).

LEMMA 3: *Under assumptions of gross substitutability* (*S*) *and homogeneity* (*h*), *for any* $P > 0$, $P \neq \lambda\bar{P}$ *for any* $\lambda > 0$,

$$P_{j'}/\bar{P}_{j'} = max_j \, P_j/\bar{P}_j, \qquad P_{j''}/\bar{P}_{j''} = min_j \, P_j/\bar{P}_j$$

implies respectively that

$$X_{j'}(P) - \bar{X}_{j'} < 0, \qquad X_{j''}(P) - \bar{X}_{j''} > 0 .$$

[24] Stability is proved in the same way as in Theorem 1 for the case of the weak axiom of revealed preference and that of "no trade" at equilibrium. See Arrow, Block, and Hurwicz [**4**].

A sketch of the proof (Arrow, Block, and Hurwicz [**4**, Lemma 3]) is given below.

Let us define

$$P^* = (\bar{P}_{j'}/P_{j'})P .$$

Then by hypothesis, $P^* \leqslant \bar{P}$ and $P^*_{j'} = \bar{P}_{j'}$. Hence, by (e), (h), and (S),

$$X_{j'}(P) = X_{j'}(P^*) < X_{j'}(\bar{P}) = \bar{X}_{j'} .$$

Similarly, with

$$P^{**} = (\bar{P}_{j''}/P_{j''})P ,$$
$$\bar{P} \leqslant P^{**}, \qquad \bar{P}_{j''} = P^{**}_{j''} ,$$

we have $X_{j''}(P) = X_{j''}(P^{**}) > X_{j''}(\bar{P}) = \bar{X}_{j''}$.

This lemma says that the commodity whose price is the maximum relative to the equilibrium price has a negative excess demand and, according to the tâtonnement (T), its price is decreasing, while the commodity whose price is the minimum has a positive excess demand and its price is increasing. In the case of two commodities, $m = 2$, the above argument can be shown graphically (see Figure 3). We have $j' = 1$, $j'' = 2$, since $P_1/\bar{P}_1 > P_2/\bar{P}_2$.

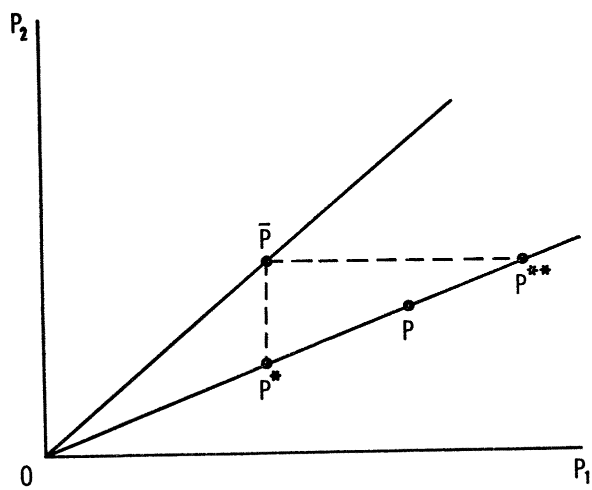

FIGURE 3

Since demand is positively homogeneous, $X_1(P) = X_1(P^*)$. Compare P^* with $\bar{P}$. While the price of the first commodity remains constant, the price of the second commodity goes up. Therefore we have

$$X_1(P) = X_1(P^*) < X_1(\bar{P}) = \bar{X}_1 .$$

Similarly we have

$$X_2(P) = X_2(P^{**}) > X_2(\bar{P}) = \bar{X}_2 .$$

By using this lemma, the stability of (T) is proved.

THEOREM 2: *Under the assumptions of homogeneity* (h) *and gross substitutability* (S) *of the demand function, the tâtonnement process* (T) *is globally stable.*[25]

A sketch of the proof (Arrow, Block, and Hurwicz [**4**]) is given below.

Let us define the distance from the variable point $P(t)$ of the solution for the tâtonnement process (T) to an equilibrium $\bar{P}$, such that

$$\Sigma_j P_j^2(0) = \Sigma_j \bar{P}_j^2 ,$$

in terms of the maximum norm as

$$D_m(t) = \max_j \{|[P_j(t) - \bar{P}_j]/\bar{P}_j|\} .$$

If this function is differentiable, we have from Lemma 3,

$$\dot{D}_m(t) = -|[X_k(P) - \bar{X}_k]/\bar{P}_k| ,$$

where k is such that

$$|P_k/\bar{P}_k - 1| = \max_j |P_j/\bar{P}_j - 1|$$

and $\dot{D}_m(t) < 0$.

With some complications due to the fact that $\dot{D}_m$ does not exist everywhere, we are able to show that $D_m(t)$ decreases through time, and the convergence of the solution $P(t)$ to an equilibrium $\bar{P}$ is established.

It should be noted that the positive homogeneity of the demand function in prices (h) plays an important role in the above argument.

In the case of two commodities, $m = 2$, the convergence is shown graphically (see Figure 4).

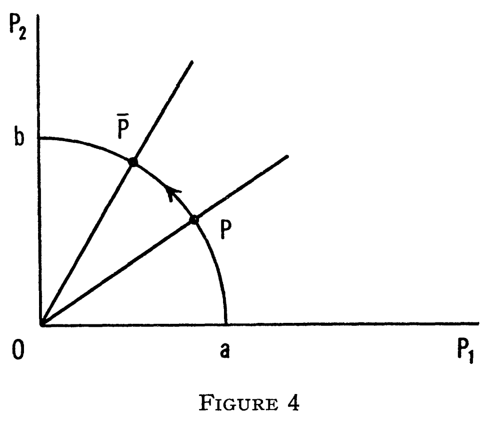

FIGURE 4

[25] We can extend this theorem for the case of nonlinear price adjustment,

$$\dot{P}_j = H_j[X_j(P) - \bar{X}_j] ,$$

where H_j is a sign preserving function such that sign (H_j) = sign $(X_j - \bar{X}_j)$. See Arrow, Block, and Hurwicz [**4**].

Remembering that $\sum_j P_j^2(t)$ is constant, we know that $P(t)$ must be on the curve ab (a part of a circle around 0). P_1 is maximum and P_2 is minimum relative to $\bar{P}$. From Lemma 3, there is an excess supply of the first commodity and an excess demand for the second commodity. Therefore, P_1 goes down and P_2 goes up, so that $P(t)$ moves in the direction of $\bar{P}$ and finally gets there.

6. A CONTRIBUTION BY ALLAIS

A brief comment may be appropriate on the significance of the rather neglected contribution to this field by M. Allais [**1**, vol. 2, pp. 486–489] in 1943.

The stability of the Walrasian tâtonnement is discussed. It must be mentioned first that the original model of the tâtonnement due to Allais is not our tâtonnement process (T) in the sense that price adjustment is assumed to take place not simultaneously in all markets but successively in one market after another (see footnote 21 above). Secondly, it must be noted that Allais did not assume gross substitutability (S) explicitly but made assumptions which, taken together, are essentially the same as the gross substitutability (S).

Admitting these points, we can reconstruct Allais' argument in terms of our own model (T) under assumptions (S) and (W).[26] It will be shown that Allais' argument, if properly reformulated, is the proof of the stability by the method of Lyapunov [**33**], i.e., by the use of a function decreasing through time.

Consider the sum of the absolute value of the excess demand of each commodity multiplied by its price,

$$D_a(t) = \sum_j |P_j(X_j - \bar{X}_j)|\,.$$

By Walras' law (W),

$$D_a(t) = 2\sum_{j\epsilon J^+} P_j(X_j - \bar{X}_j) = -2\sum_{j\epsilon J^-} P_j(X_j - \bar{X}_j)\,,$$

where J^+ is the set of those j's for which $X_j - \bar{X}_j > 0$, and J^- is the set of those j's for which $X_j - \bar{X}_j < 0$. When the derivative of $D_a(t)$ exists,

$$\begin{aligned}\dot{D}_a(t) &= \sum_{k\epsilon K^+} \frac{\partial \sum_j |P_j(X_j - \bar{X}_j)|}{\partial P_k}\frac{dP_k}{dt} + \sum_{k\epsilon K^-} \frac{\partial \sum_j |P_j(X_j - \bar{X}_j)|}{\partial P_k}\frac{dP_k}{dt} \\ &= 2\sum_{k\epsilon K^+} \frac{\partial\{-\sum_{j\epsilon J^-} P_j(X_j - \bar{X}_j)\}}{\partial P_k}\dot{P}_k + 2\sum_{k\epsilon K^-} \frac{\partial\{\sum_{j\epsilon J^+} P_j(X_j - \bar{X}_j)\}}{\partial P_k}\dot{P}_k\end{aligned}$$

where K^+, K^- are defined similarly as J^+, J^-, i.e., K^+ is the set of those k's for which $X_k - \bar{X}_k > 0$ and K^- is the set of those k's for which $X_k - \bar{X}_k < 0$.

[26] A more rigorous discussion under weaker assumptions is given in McKenzie [**37**]. See also Negishi [**45**].

Because K^+ and J^-, K^- and J^+ are non-overlapping, respectively, we have $j \neq k$ in the above expression of $\dot{D}_a(t)$ and by (S), $\partial X_j/\partial P_k > 0$. Then we have $\dot{D}_a(t) < 0$ at disequilibrium, since $\dot{P}_k > 0$ for $k \varepsilon K^+$ and $\dot{P}_k < 0$ for $k \in K^-$ from the construction of the process (T).

With some complications due to the fact that $\dot{D}_a(t)$ does not exist everywhere, we are able to show that $\dot{D}_a(t)$ decreases along with the solution of (T) and $P(t)$ converges to equilibrium.

7. EXAMPLES OF INSTABILITY (SCARF)

In the preceeding sections, an example of a stable tâtonnement process is shown. Is the tâtonnement process generally stable without any restrictions such as gross substitutability? The answer is no, because we have an example of instability due to Scarf [**55**].

Consider the case of three individuals, $n = 3$, and three commodities, $m = 3$. Let the utility function of the first, second, and third individuals, respectively, be

$$U_1(X_{11}, X_{12}, X_{13}) = \min (X_{11}, X_{12}) ,$$
$$U_2(X_{21}, X_{22}, X_{23}) = \min (X_{22}, X_{23}) ,$$
$$U_3(X_{31}, X_{32}, X_{33}) = \min (X_{33}, X_{31}) ,$$

and the holding of commodities be

$$\bar{X}_{ij} = 1 \qquad \text{for } i = j ,$$
$$\bar{X}_{ij} = 0 \qquad \text{for } i \neq j .$$

Each individual desires only two commodities, which are perfectly (intrinsically) complementary, i.e., desired only in the fixed ratio (one to one in this case). For example, the indifference curves of the first individual for the first two commodities are of the form shown in Figure 5, and the first

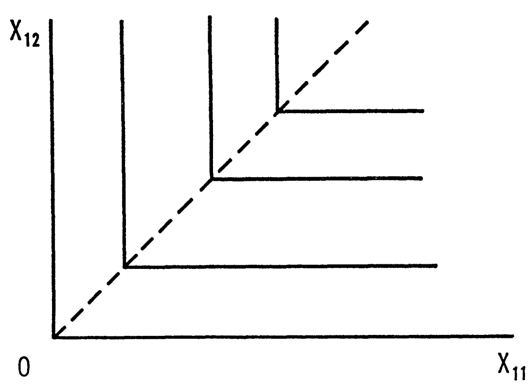

FIGURE 5

individual has no desire for the third commodity. Routine calculations show that the excess demand for each commodity is,

$$X_1 - \bar{X}_1 = \frac{-P_2}{P_1 + P_2} + \frac{P_3}{P_3 + P_1},$$

$$X_2 - \bar{X}_2 = \frac{-P_3}{P_2 + P_3} + \frac{P_1}{P_1 + P_2},$$

$$X_3 - \bar{X}_3 = \frac{-P_1}{P_3 + P_1} + \frac{P_2}{P_2 + P_3},$$

and that $P_1 = P_2 = P_3$ is the only equilibrium possible.

Consider a solution $P(t)$ of the tâtonnement process (T) with initial prices $(P_1(0), P_2(0), P_3(0))$ such that

$$P_1^2(0) + P_2^2(0) + P_3^2(0) = 3\,,$$

and

$$P_1(0)\,P_2(0)\,P_3(0) \neq 1\,.$$

From Walras's identity, we know that $\sum_j P_j^2(t) = 3$ for $t \geqslant 0$ and the only possible equilibrium is (1, 1, 1). On the other hand, by differentiating with respect to time, it can be shown that

$$P_1(t)\,P_2(t)\,P_3(t) = P_1(0)\,P_2(0)\,P_3(0)$$

for $t \geqslant 0$, and $P(t)$ never reaches equilibrium.

In this example, since $\sum_j P_j^2(t) = \sum_j P_j^2(0)$, prices still remain close to equilibrium if initially they are sufficiently close to it. Therefore, stability in the sense of Lyapunov (Section 2.3) is obtained although global stability is not established.

It is known that the stability condition is satisfied if asymmetrical income effects are neglected.[27] In this example, on the contrary, substitution effects do not exist and income effects are dominating. It has been long conjectured that instability, if any, might be due to the income effect, and in Scarf's example this is exactly the case.

One might object to the special properties of the above mentioned example. An example of a more general type is also given in Scarf [**55**]. Though it is difficult to characterize precisely those markets that are unstable, it is rather clear that instability is a relatively common phenomenon.

Judging from these examples, we must admit that the tâtonnement process (T) is not perfectly reliable as a computing device to solve the system of equations for general economic equilibrium (e). It is possible to

[27] For the relation of stability to the substitution and income effects of price changes, see Hicks [**26**, p. 317]; Samuelson [**54**, p. 270]; and Arrow and Hurwicz [**6**].

interprete these instability examples as showing that the difficulty is essentially due to the assumption of tâtonnement (no trade out of equilibrium) and to conclude that the tâtonnement process (T) does not provide a correct representation of the dynamics of markets. See Hahn [**23**a].

8. NON-TÂTONNEMENT PROCESSES

The failure of the general stability of the tâtonnement process (T) suggests the study of the stability of the non-tâtonnement processes (NT) (Section 4.2).

Let us assume as a transaction rule that no transaction on credit is permitted, i.e., all transactions should be of the barter type in the non-tâtonnement process. Since in a barter exchange, to get something one must offer something else of the same value in return, such an exchange does not alter the value of the commodity stocks held by an individual, i.e., his income. It may therefore be appropriate to assume the following restrictions on the functions F (see Negishi [**45**]).

$$\text{(B)} \qquad \Sigma_j P_j \dot{\bar{X}}_{ij} = \Sigma_j P_j F_{ij}(P, \bar{X}) = 0 \qquad (i = 1, \ldots, n).$$

In this section it will be shown that any non-tâtonnement process,

$$\text{(NT)} \qquad \begin{aligned} \frac{dP_j}{dt} &= X_j(P, \bar{X}) - \bar{X}_j & (j = 1, \ldots, m), \\ \frac{d\bar{X}_{ij}}{dt} &= F_{ij}(P, \bar{X}) & (j = 1, \ldots, m;\ i = 1, \ldots, n), \end{aligned}$$

which satisfies the condition of barter exchange (B) is quasi-stable under gross substitutability (S) and Walras' law (W). The stability of more special non-tâtonnement processes with additional conditions imposed on transaction rules will be discussed in Sections 9 and 10.

In Section 6, the stability of the tâtonnement process is shown, following the method of Allais [**1**], by using the fact that

$$D_a(t) = \Sigma_j | P_j(X_j - \bar{X}_j) |$$

is decreasing along with the solution of the process. This is still true in the present case. Let us use the same $D_a(t)$ in the case of the non-tâtonnement process (NT). Suppose $\dot{D}_a(t)$ exists. In this case we have,

$$\dot{D}_a(t) = \Sigma_k \frac{\partial \Sigma_j | P_j(X_j - \bar{X}_j) |}{\partial P_k} \dot{P}_k + \Sigma_i \Sigma_k \frac{\partial \Sigma_j | P_j(X_j - \bar{X}_j) |}{\partial \bar{X}_{ik}} \dot{\bar{X}}_{ik}.$$

As in Section 6, the first term of the right hand side is negative at disequilibrium while the second term can be expressed as

$$\Sigma_i \Sigma_k \frac{\partial \Sigma_j | P_j(X_j - \bar{X}_j) |}{\partial I_i} \frac{\partial I_i}{\partial \bar{X}_{ik}} \dot{\bar{X}}_{ik}$$

since the dependence of X_j on $\bar{X}_{ik}$ is through the income of the ith individual, I_i.[28] By definition $I_i = \Sigma_j P_j \bar{X}_{ij}$, and so $\partial I_i / \partial \bar{X}_{ik} = P_k$. Since in (NT) $\dot{\bar{X}}_{ij} = F_{ik}(P, \bar{X})$, we have

$$\Sigma_k \frac{\partial I_i}{\partial \bar{X}_{ik}} \dot{\bar{X}}_{ik} = \Sigma_k P_k F_{ik} = 0$$

from the condition of barter (B). Therefore, $\dot{D}_a(t) < 0$ again in this case.

With some complications due to the fact that $\dot{D}_a(t)$ does not exist everywhere, we are able to show that $D_a(t)$ decreases through time and to establish the quasi-stability of a non-tâtonnement process under assumptions (B), (S), (W) (see Negishi [**45**]). Global stability is, however, not established since nothing is known in this case about the uniqueness of the equilibrium prices and the distribution which satisfy condition (e′) in Section 4.2.

9. EDGEWORTH PROCESS

9.1. In the previous sections, the non-tâtonnement process (NT) as well as the tâtonnement process (T) was shown to be stable under gross substitutability (S). So far, however, little has been assumed for the non-tâtonnement (NT) process about how and why individuals exchange their stocks of commodities. It is expected that by adding some plausible assumptions on the exchange behavior of individuals, stronger results on stability may be established.

Let us consider why individual participants exchange their stocks of commodities. Except for the purpose of speculation, people want to exchange so as to increase their satisfactions. It is rather plausible, therefore, to assume that, in the non-tâtonnement process, the utility of the commodity stock for each individual increases through time by virtue of the exchange transactions.

The Edgeworth process, so named by Uzawa [**59**], is defined as a non-tâtonnement process (NT) with the following transaction rule (E):

(E) $dU_i(\bar{X}_{i1}, \ldots, \bar{X}_{im})/dt \geqslant 0$ for all i, and $dU_i(\bar{X}_{i1}, \ldots, \bar{X}_{im})/dt > 0$ for some i, if there is a distribution $\bar{X}'$ such that $U_i(\bar{X}'_{i1}, \ldots, \bar{X}'_{im}) \geqslant U_i(\bar{X}_{i1}, \ldots, \bar{X}_{im})$, $\Sigma_j P_j \bar{X}'_{ij} = \Sigma_j P_j \bar{X}_{ij}$, for all i, and $U_i(\bar{X}'_{i1}, \ldots, \bar{X}'_{im}) > U_i(\bar{X}_{i1}, \ldots, \bar{X}_{im})$, for some i; otherwise, $dX_{ij}/dt = 0$ for all i, j.

The condition (E) states that trade will take place if and only if at least one individual gains by exchange and no individual loses. The distribution of commodities $\bar{X}'$ is clearly more satisfactory for somebody and less satisfactory for nobody than the original distribution $\bar{X}$. Moreover, $\bar{X}'$ can

[28] That effects of transactions at disequilibria are income effects is stated in Hicks [**26**, p. 128].

be reached from $\bar{X}$ by the barter of commodities, keeping budget constraints for individuals satisfied.

An example of the Edgeworth process, (NT) with (C), (B) and (E) satisfied, is given below (Uzawa [**59**]).

Let us define a social utility function as a weighted sum of individual utility functions,

$$U = \Sigma_i c_i U_i(X_{i1}, \ldots, X_{im}) ,$$

where c_i is a positive constant. Let $\hat{X}_{ij}$ be the solution for the maximization of the social utility subject to

$$\Sigma_i X_{ij} = \Sigma_i \bar{X}_{ij} ,$$

$$\Sigma_j P_j X_{ij} = \Sigma_j P_j \bar{X}_{ij} ,$$

and

$$U_i(X_{i1}, \ldots, X_{im}) \geqslant U_i(\bar{X}_{i1}, \ldots, \bar{X}_{im}) \text{ for all } i, j.$$

At the distribution $\hat{X}$, everybody is clearly better off than at the distribution $\bar{X}$. The individual's weight, c_i, in the above social utility function may be interpreted as an index of his bargaining power in the exchange of commodities.

Consider the process given by

$$\text{(NTE)} \qquad \begin{aligned} \frac{dP_j}{dt} &= X_j - \bar{X}_j , \\ \frac{d\bar{X}_{ij}}{dt} &= \hat{X}_{ij} - \bar{X}_{ij} , \qquad \text{for all } i, j. \end{aligned}$$

This implies that prices change in accord with excess demands while trades take place among individuals, starting from the distribution $\bar{X}$ and directed to the distribution $\hat{X}$. It is easily seen that (NTE) satisfies condition (C), (B), and (E).

Condition (C) implies that every solution of the process (NTE) is bounded, while from the assumption (E), each individual's utility increases along the path. It follows, as has been shown by Uzawa [**59**], that the Edgeworth process (NTE) is globally stable provided certain additional assumptions are satisfied.[29]

9.2. It is observed that for the Edgeworth process (E), if there is no trade possible among individuals, i.e., $\dot{\bar{X}}_{ij} = 0$ for all i, j, then there must exist

[29] The proofs of the stability of the Edgeworth process, as well as those of the Hahn process in Section 10, are not directly dependent on the assumptions (W) and (h). However, one may regard these assumptions as the rationale underlying the formulation of the processes.

a commodity r which has the maximum marginal utility at $\bar{X}$ relative to its price for all individuals:

$$\left[\frac{\partial U_i(\bar{X}_{i1},\ldots,\bar{X}_{im})}{\partial \bar{X}_{ir}}\Big/P_r\right] = \max_j \left[\frac{\partial U_i(\bar{X}_{i1},\ldots,\bar{X}_{im})}{\partial \bar{X}_{ij}}\Big/P_j\right]$$

for all i.

If this is not true, i.e., if for different individuals a different commodity has the maximum relative utility, then there is possible a gain for some individuals without causing a loss for any individual, each individual receiving a commodity with the maximum relative utility for him in exchange for other commodities, and trade among individuals must take place according to (E).

Let us assume that for any such commodity r which has the maximum relative utility for all individuals, excess demand $X_r(P, \bar{X}) - \bar{X}_r$ is positive. This assumption is satisfied if the utility functions are of additive form:

$$U_i(X_{i1},\ldots,X_{im}) = U_{i1}(X_{i1}) + \ldots + U_{im}(X_{im}) \;;$$

but it may be violated if there is a strong substitutability in the sense of Edgeworth and Pareto[30] among commodities.[31]

By this assumption it is shown in the Edgeworth process that the function

$$\frac{\partial U_i(\bar{X}_{i1},\ldots,\bar{X}_{im})}{\partial \bar{X}_{ir}}\Big/P_r$$

is decreasing over time when no trade is possible, i.e., $\bar{X}$ is constant, since its derivative with respect to time is,

$$-\frac{\partial U_i(\bar{X}_{i1},\ldots,\bar{X}_{im})}{\partial \bar{X}_{ir}}\,\frac{(X_r - \bar{X}_r)}{P_r^2} < 0\,.$$

Using this fact, together with the increase of $\sum_i U_i(\bar{X}_{i1},\ldots,\bar{X}_{im})$ when trade is possible, the quasi-stability of the Edgeworth process is demonstrated by Hahn [**23**].

9.3. In the case of two individuals and two commodities, $m = 2$, $n = 2$, we can show the path of an Edgeworth process in a box diagram (Figure 6)

[30] Commodities j and k are substitutes in the sense of Edgeworth-Pareto if $\partial^2 U/\partial X_j \partial X_k < 0$. See Hicks [**26** p. 42].

[31] We are then in a peculiar situation where everybody wants to have more of the stock of commodity r with the maximum relative utility while the excess demand for that commodity is negative and the price falls. This may suggest the inappropriateness, in the case of the Edgeworth process, of the assumption that prices change according to excess demands in the usual sense.

where the commodity stocks of the first individual are measured from A, and those of the second, from B. $\bar{X}$ is the point corresponding to the distribution of commodities $\bar{X}$; curves $\bar{X}c$ and $\bar{X}d$ are indifference curves of the first and second individuals, respectively, through $\bar{X}$; and $\bar{X}e$ is a solution of the Edgeworth process. Condition (E) states that every solution remains in the shaded area[32] and terminates on the contract curve ab.[33] Lines $\bar{X}P^1$ and $\bar{X}P^2$ are price lines which represent budget constraints of both individuals and show the possible directions of exchanges. If the price line through $\bar{X}$ is always directed into the shaded area as is $\bar{X}P^1$, the stability of the process is self evident. If not (the case of $\bar{X}P^2$), we have to assume conditions as in Sections 9.2 and 9.1 that will push the price line into the shaded area.

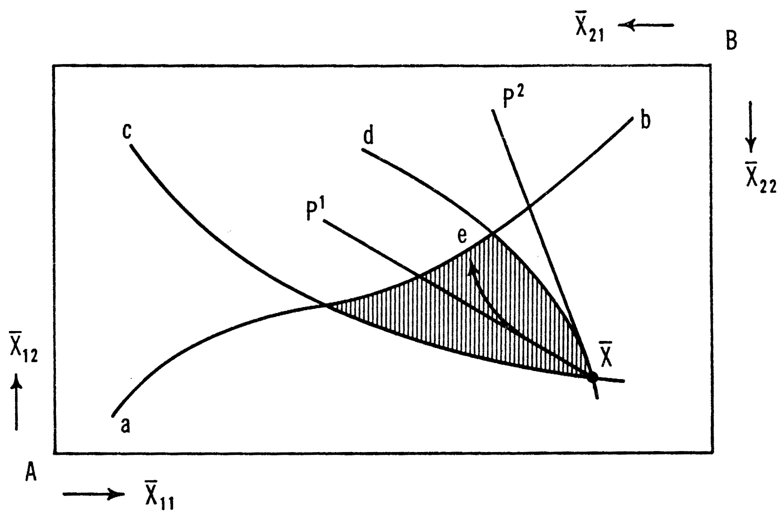

FIGURE 6

10. THE HAHN PROCESS

10.1. In the non-tâtonnement process (NT), it is assumed that the opportunity for an individual to change his stock of commodities through exchange depends generally on prices and the distribution of commodities among individuals. In other words, the individual's opportunity to change his commodity stock depends on the plans of other individuals to change theirs. If there is a surplus of a commodity, i.e., if the total amount individuals want to increase their stock of this commodity is less than the amount they want to part with, any individual who seeks to increase his stock can easily

[32] I.e., the set of distributions at which everybody is not worse off than at the distribution $\bar{X}$.

[33] The set of Pareto-optimal distributions, or, in other words, points where the indifference curves of two individuals are tangent one to another.

and quickly achieve his plan. If there exists, on the other hand, a shortage of a commodity, any individual who wants to dispose of his stock can easily and quickly do so.

The Hahn process (Hahn [**22**], Hahn and Negishi [**24**]) is a non-tâtonnement process (NT) with the following conditions imposed on functions F in addition to (C) and (B):

(H) For all i, j, sign $(X_{ij} - \bar{X}_{ij}) =$ sign $(X_j - \bar{X}_j)$ if $X_{ij} - \bar{X}_{ij} \neq 0$; and if $X_j - \bar{X}_j = 0$, then $X_{ij} - \bar{X}_{ij} = 0$ for all i.

The implication of condition (H) is this: With prices given, all possible exchanges are instantaneously effected on a "first come, first served" basis. Thus, if $X_j - \bar{X}_j < 0$, all individuals demanding the jth commodity will be able to satisfy their demand, while some supplying individuals will be left with unsold goods. Therefore, after exchange, there remain only negative individual excess demands (positive excess supplies) and sign $(X_{ij} - \bar{X}_{ij}) =$ sign $(X_j - \bar{X}_j) < 0$. If $X_j - \bar{X}_j > 0$, then all supplying individuals will find they can supply all they had planned, while some demanding individuals will have their demands unsatisfied, with the result that sign $(X_{ij} - \bar{X}_{ij}) =$ sign $(X_j - \bar{X}_j)$.

The stability of the Hahn process, a non-tâtonnement process (NT) with conditions (B), (C), and (H) satisfied, is proved in the following way.

Differentiating the budget constraint of the ith individual, we have

$$\Sigma_j P_j(\dot{X}_{ij} - \dot{\bar{X}}_{ij}) + \Sigma_j \dot{P}_j(X_{ij} - \bar{X}_{ij}) = 0 .$$

From condition (B), $\Sigma_j P_j \dot{\bar{X}}_{ij} = 0$, and since from (NT) and (H), $\Sigma_j \dot{P}_j (X_{ij} - \bar{X}_{ij}) > 0$, we have $\Sigma_j P_j \dot{X}_{ij} < 0$ in disequilibrium. On the other hand, consider $\Sigma_i U_i(X_{i1},\ldots, X_{im})$. By differentiation with respect to time, we have in disequilibrium

$$\Sigma_i \Sigma_j \frac{\partial U_i}{\partial X_{ij}} \dot{X}_{ij} = \Sigma_i \lambda_i \Sigma_j P_j \dot{X}_{ij} < 0 ,$$

where λ_i is the marginal utility of income, a positive function of t. From the decrease of the sum of utilities in disequilibrium, we can infer quasi-stability (See Hahn and Negishi [**24**]).

10.2. The question of the existence of a Hahn process, that is, the existence of a solution of (NT) under (C), (B), and (H) remains still open. We hope it is answered partly by the following example.

Let us construct an example of a Hahn process where the excess or the shortage of commodities is always shared by individuals in constant ratio:

$$\text{(NTH)} \qquad \frac{dP_j}{dt} = X_j - \bar{X}_j \qquad (j = 1, \ldots, m) \ ;$$

$$\frac{d\bar{X}_{ij}}{d\,t} = \Sigma_k \frac{\partial X_{ij}}{\partial P_k}(X_k - \bar{X}_k) - \alpha_i \Sigma_k \Sigma_s \frac{\partial X_{sj}}{\partial P_k}(X_k - \bar{X}_k)$$

$$(j = 1, \ldots, m-1\,;\ i = 1, \ldots, n) \ ;$$

$$\frac{d\bar{X}_{im}}{dt} = \frac{1}{P_m}\Big(-\sum_{j=1}^{m-1} P_j \dot{\bar{X}}_{ij}\Big) \qquad (i = 1, \ldots, n) \ ;$$

where the α_i's are positive constants such that $\Sigma_i \alpha_i = 1$, i.e., the ratio by which an excess or shortage of commodities is allotted to individuals.

It is easily verified that this process satisfies conditions (C) and (B). Then we know from (B) that

$$\frac{dX_{ij}}{dt} = \Sigma_k \frac{\partial X_{ij}}{\partial P_k}\dot{P}_k + \Sigma_k \frac{\partial X_{ij}}{\partial \bar{X}_{ik}}\dot{\bar{X}}_{ik} = \Sigma_k \frac{\partial X_{ij}}{\partial P_k}(X_k - \bar{X}_k) \, ,$$

since

$$\Sigma_k \frac{\partial X_{ij}}{\partial \bar{X}_{ik}}\dot{\bar{X}}_{ik} = \Sigma_k \frac{\partial X_{ij}}{\partial I_i}\frac{\partial I_i}{\partial \bar{X}_{ik}}\dot{\bar{X}}_{ik} = \frac{\partial X_{ij}}{\partial I_i}\Sigma_k P_k \dot{\bar{X}}_{ik} = 0 \, ,$$

and from (NTH) we have

$$\dot{\bar{X}}_{ij} = \dot{X}_{ij} - \alpha_i \Sigma_i \dot{X}_{ij}, \qquad j \neq m \, .$$

Let us postulate that prices remain constant in the small interval of time $(0, h)$, and at $t = h$, as a result of competitive exchange in the interval, we have

$$X_{ij}(h) - \bar{X}_{ij}(h) = \alpha_i(\Sigma_i X_{ij}(h) - \bar{X}_j) \, , \qquad \text{for all } j \, .$$

Then we have

$$X_{ij}(t) - \bar{X}_{ij} = \alpha_i(\Sigma_i X_{ij}(t) - \bar{X}_j) \qquad \text{for all } i, j \neq m; t \geqslant h \ ;$$

and from Walras's law (W) and individual budget constraints,

$$X_{im}(t) - \bar{X}_{im}(t) = \alpha_i(\Sigma_i X_{im}(t) - \bar{X}_m) \, ,$$

for all i, and for $t \geqslant h$, provided $P > 0$, $\bar{X} > 0$.

Thus (NTH) satisfies condition (H) (Hahn and Negishi [**24**]).

11. CONCLUDING REMARKS — SUGGESTIONS FOR FURTHER STUDIES

To conclude this survey, let me state a number of problems which have remained rather untouched so far but should be studied extensively hereafter.

As is mentioned in Section 4.2 and footnotes 7 and 31, the process of price change in accord with excess demand may not be generally correct except in the case of tâtonnement in a perfectly organized market. It would seem worthwhile to construct various models of a price formation process for the less organized markets, i.e., a series of bargains among individuals with or without recontract. Study of the non-competitive price formation process with price leadership or full-cost-pricing may also be interesting.

The price formation process can be extended over Hicksian weeks. We may cite, for example, the cobweb process, the process with interactions between expectations and inventory fluctuations, and the Marshallian dynamic process (Section 2.1). The stability of these processes, as well as of the magnificent dynamic processes cited in Section 2.1, might well be analysed by methods and tools developed in the study of tâtonnement and non-tâtonnement stability.

The dynamic model analysed in this article has been of differential or continuous-time type. Models of difference or discrete-time type have also been studied. Models of these two types are simple and convenient from the mathematical point of view. The introduction of lagged adjustments (Section 2.1), however, will sometimes require the study of general mixed, discrete and continuous, time models.

We discussed in Section 2.2 the need to study stability. It should be mentioned that the various reasons given there suggest the need not only for stability, i.e., convergence to equilibrium, but also for quick convergence to equilibrium. The speed with which the system converges to equilibrium must also be estimated.[34]

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