Available online at www.sciencedirect.com

ScienceDirect

Discrete Optimization 5 (2008) 159–167

DISCRETE OPTIMIZATION

www.elsevier.com/locate/disopt

George Dantzig in the development of economic analysis

Kenneth J. Arrow*

Department of Economics, Stanford University, Stanford, CA 94305-6072, USA

Received 16 January 2006; accepted 20 November 2006
Available online 25 October 2007

Abstract

The role of optimization is central to economic analysis, particularly in its "neoclassical" phase, since about 1870, and is therefore highly compatible with the impulse behind linear programming (LP), as developed by Dantzig. LP's stress on alternative activities fits very well with modern economic analysis. The concept of economic equilibrium, properly understood, required the central notion of complementary slackness. so central in LP.

LP was seen as a tool for actual implementation of neoclassical principles precisely at a time when the market was under attack from several directions. The economists Koopmans and Hurwicz played an important role both in stimulating the crucial development of the simplex method and in relating LP to the world of economics.

LP became widely used in national economic planning, particularly for developing countries, and for the study of individual industries, especially the energy sector. The works of Chenery and of Manne are central in these fields.

As respect for the usefulness of the market increased, the emphasis on national planning diminished and was replaced by an emphasis on equilibrium analysis, in which LP still plays a large part in the study of individual sectors, particularly energy.

Keywords: Linear programming; George B. Dantzig; Economic planning; History of economic thought; History of operations research

1. Introduction

Economic analysis, almost since its inception, has been concerned with optimization, and that in a dual sense. For one thing, it tends to presuppose that the individual members of the economy are optimizing. Exactly what they are optimizing and what are the constraints differ from case to case. For another thing, economic analysis is designed not only for descriptive study of the economy but also for recommending policy, that is, for improving or (ideally) optimizing the outcome of the economy. The second sometimes takes the form of theorems stating that, under certain hypotheses, the spontaneous actions of the economy result in an optimum. Sometimes instead it is asserted that government policies (taxes, tariffs, price regulations, government expenditures) are needed for optimization or improvement.

Since linear programming is concerned above all with optimization under constraints, albeit under a special set of conditions, it is not surprising that linear programming has been closely related to economics and has emerged as a major instrumentality for certain purposes. Much of this connection is illustrated in George Dantzig's work, both in the development of the simplex method and in its adoption as a major tool of economic model building.

* Tel.: +1 650 327 3957; fax: +1 650 326 1702.
E-mail address: arrow@stanford.edu.

1572-5286/$ - see front matter
doi:10.1016/j.disopt.2006.11.007

2. Production and optimization

The chief agents in the economy are the consumers and the producers. In modern economic analysis (from about 1870), both are assumed to be optimizing. However, it is the optimization on the part of the producers that has in many ways dominated the analysis and is closer to linear programming.

The characteristic assumption is that of perfect competition, though broader concepts (e.g. monopoly) appeared already in the first half of the nineteenth century [13, Chapter 5], [43, Book III, Chapter II, section 5; first edition, 1848]. Under competition, the individual firm (or other producing unit) takes the prices at which it buys inputs and sells output as given, so that its payoff is a linear function of output and inputs. (Monopoly would imply that the payoff function is nonlinear.) The constraints are defined by the different methods of producing the output from the inputs and by the resources that are given to the firm and not purchasable on the market. An early example is Ricardo's analysis of farming [47, Vol. I, Chapter II; originally published in 1817]. The farmer has a given amount of land available. He can hire labour at a fixed wage; Ricardo assumed that additional units of labour have diminishing productivity. Then, given the price of the product and the wage rate, there is an optimal amount of labour. Ricardo considered more generally the case where different pieces of land have differing productivities. In general, some pieces of land will not be used, i.e. will be slack variables in LP terms.

This emphasis on choice among alternative ways of producing a single good was not universally applied. Ricardo's description of industrial production was less well developed than that of agriculture, but implicitly, at least, there seemed to be only one way of producing a given industrial good. This hypothesis was set forth explicitly by Walras in the first edition of his work (1871; see [51]). Once data were available on the sales and purchases between industry, this constant-coefficients hypothesis became (much later) the theoretical basis for Leontief's input–output analysis [30]. Although the absence of alternative processes hardly pointed to the value of linear programming in choosing among alternatives, input–output analysis exhibited activities as vectors and supplied the data which helped make possible the application of linear programming to national economies.

The analogue of Ricardo's analysis of agriculture began to be developed for industry in general in the form of the *production function*, which relates the output of an industry to its inputs. In particular, this formulation implies that different combinations of inputs can yield the same output. This concept appeared in the 1880s; see [9] and in the later editions of Walras's book (cited above). As in Ricardo's analysis of agriculture, the firm maximized profit, which was a linear function of output and inputs. If the production function is written as

$$y = F(x), \tag{1}$$

where $y =$ output, and x is the vector of inputs; if we let p be the price of output and q the vector of input prices, then profit (Π) is given by

$$\Pi = pF(x) - qx \tag{2}$$

and optimization entails the obvious conditions that

$$F_x = q/p \tag{3}$$

provided that, at the optimum, $x_i > 0$, all i.

Suppose we interpret the production function as applying to the entire economy, and suppose further that we imagine an economy with only one output and with inputs as given in magnitude. That is, the inputs are themselves not produced but are ultimate factors of production, such as land, labour and capital (the latter is indeed produced, but at any moment of time the amount is given by past decisions). Let ξ be the vector of initial supplies of inputs.

Suppose (3) has a solution for each set of possible values of the input price vector q, $x = x(q)$. Then for feasibility we must have

$$x(q) = \xi,$$

so that q is determined by the production function, F and the supply of inputs, ξ. It is now simple to see that the vector q can be considered as a vector of Lagrange multipliers (or dual variables or shadow prices). Consider the problem of maximizing the value of output subject to the constraints, $x = \xi$. Then we get exactly the relations (3). A component, q_i represents the unit value of an infinitesimal variation in the amount of the corresponding input, ξ_i.

Here the optimization of output for a given input vector was trivial, since output is determined by inputs. However, a simple generalization makes clear the allocative role of the input price vector. Suppose there are many outputs y_i, with corresponding prices p_i, taken as given. Let $F^i(x^i)$ be the output of commodity i when the ith industry is allocated a vector of inputs, x^i. Then maximizing the value of production requires maximizing $\Sigma p_i F_i(x^i)$ subject to the constraints, $\Sigma x^i = \xi$. Again this procedure will lead to a vector, q, of Lagrange multipliers, which can be interpreted as the prices of the inputs. Notice that each industry, taking input prices as given, will maximize its profits, while in the aggregate, demand for the input will equal the given supply. This optimality of the market is part of what Adam Smith called, “the invisible hand,” generally referred to today as the basic theorem of welfare economics.

The assumption that every input is a factor in fixed supply is made here only for expository simplification. Individual firms may use the outputs of other firms as inputs, along with primary factors.

One of the assumptions made in the development of the theory of production was that the production function (or industry production functions) displayed what was called, “constant returns to scale,” or, mathematically, that the function F was homogeneous of degree 1. The argument was the standard one for linear programming; if an activity can be carried on, it can be carried on at any scale, with inputs and outputs changed in the same proportion. Under this assumption, it was early shown [52], that factor payments, $q\xi$, must necessarily equal revenues, $pF(\xi)$, or, in the many-output case, $\Sigma p_i F^i(x^i)$. The identity with the equality of the values of the primal and dual problems in linear programming is evident.

The production function concept was much discussed in the literature, and indeed some limitations pointed out, in particular, that there may be limits to the ability to substitute one input for another while maintaining output. But there was little or no empirical implementation until the very important work of Cobb and Douglas [10]. They exploited not only the production function itself but also its dual characteristics as developed above. Their application was to the national economy, so they assumed there was just one output and two inputs, capital (K) and labour (L). They used the particular form of a power function,

$$F(K, L) = AK^{\alpha}L^{1-\alpha}. \tag{4}$$

They fitted this relation to data. But they also noted that, from (3) and (4), it follows that the shares of capital and labour in total output are α and $1 - \alpha$, respectively. It turned out that the shares of capital and labour were roughly constant and at a value for which (4) fitted the data pretty well. This result started a long series of successive improvements.

3. Equilibrium and complementary slackness

In general competitive equilibrium (a concept which goes back in explicit form to Walras (1874, 1877); see [51]), prices are given to the individual firms and other economic agents such as consumers. Thus the choices of inputs and outputs by individual firms are functions of those prices. By summing over all firms and consumers, the total demand for any input is a function of prices. There was a concept that, if the supplies of the inputs are given to the economy, prices are determined by the conditions that supply and demand be equal for all factors of production.

It was always understood, though informally, that the last statement could not be true literally. There were some factors of production, e.g. air, and, in many localities, water, which were so abundant that an addition would not increase production. In those cases, supply exceeded demand. A full recognition and formal restatement first appeared in the work of Zeuthen [54]. The equilibrium condition on a factor market was restated: supply must be at least equal to demand; if greater, then the factor price was zero. Thus, the principle of complementary slackness was first given explicit notice. It played a basic role in the following literature on the consistency of the equations and inequalities of competitive equilibrium [49,50,42,1].

Most current economy-wide models are equilibrium models, rather than linear programming models, but they incorporate large elements of linear programming within them.

4. The development of linear programming and the role of economists

It may therefore be understood that at least some economists were receptive to a “bottom-up” approach to optimization. From an empirical viewpoint, it meant using a direct knowledge of the technology instead of simply inferring it from a black box in which only outputs and inputs were observable. Leontief encouraged his students to use technological data to derive industry production functions; for (nonlinear) examples, see [4]. This role has been decisive in many of the most fruitful applications of linear programming to economics.

A second point in the acceptance of and indeed enthusiasm for linear programming was that it gave directions as to what policies to take (that is, what activity levels to use) to achieve an optimum. The traditional doctrine that the market would automatically yield an efficient outcome was at that time under considerable criticism from several directions. (1) It had long been agreed that the efficiency of the market held only for competitive markets; there was considerable empirical basis for the view that monopoly power was steadily increasing. (2) The efficiency of the market also depended on the absence of what came to be known as, "externalities", cases where actions by one individual or a firm impinged on others without the intervention of the market, as with environmental effects. The importance of external effects had especially been emphasized in the widely-known work of Pigou [46]. (3) Collateral to these discussions had been a controversy over the feasibility and efficiency of a socialist economic system. Some economists opposed to socialism, most notably [44,45] had held that a socialist economy could not be viable or at any rate would be very inefficient. The so-called "market socialists" recognized the efficiency of competitive equilibrium but argued, (a) that the actual capitalist economy was far from efficient for reasons (1) and (2) above, (b) the capitalist economy yielded a very unequal distribution of income, and (c) a socialist economy could be devised which would achieve efficiency by specifying appropriate operating rules to the planning agencies and individual socialist firms. These propositions had been advanced by many authors. An early paper was that of Enrico Barone in 1906; for an English translation, see [2]. There were many papers on this subject during the period between the two World Wars but perhaps most notably that of Lange [28]. (4) The Great Depression had undermined any entire reliance on the unconstrained market. (5) World War II created an opportunity for rethinking military and related issues in accordance with principles of rationality, i.e. optimizing. This tendency became known as, "operations research." It created a demand for applying economic reasoning to the solution of highly specific and narrow problems, as opposed to the broader questions with which economists had traditionally grappled.

For all these reasons, an approach which permitted the solution of well-posed optimizing questions in specific numerical terms resonated well with at least some economists. Tjalling Koopmans was one such. As a graduate student in physics in the Netherlands, he wrote a highly-regarded paper on calculating quantum wave functions. But his social concerns led him into economics, originally into the nascent field of econometrics, that is, the development of statistical methods appropriate to the special problems of economics. He participated in the major study of business cycles conducted at the League of Nations under the leadership of another Dutch economist, Jan Tinbergen. A more junior associate in this group was Leonid Hurwicz, a young Polish economist with mathematical interests. As France fell to the German advance in World War II, both Koopmans and Hurwicz managed to get on some of the last ships to leave for the United States (in 1940). Koopmans was, for some time, associated with the (British-American) Combined Shipping Adjustment Board and with the British Merchant Shipping Mission in Washington. His duties were apparently primarily as a statistician, but he became aware of the resource allocation problem which later became known as the transportation problem. Given a number of ports and, for each pair of ports, an amount to be shipped from one to another, the aim is to minimize the number of ships needed. He developed an effective algorithm to solve this problem, though, from his account, it does not seem to have been used. His work was not published at the time. An informal account is given in [23] and the complete results are presented in [26]. (Actually, a similar solution had already been obtained by Hitchcock [19], unknown to Koopmans.)

Dantzig, in his initial and classic presentation of the simplex method [14] tells us (fn. 1) that, "The general nature of the 'simplex' approach ... was stimulated by discussions with Leonid Hurwicz. The author is indebted to T. C. Koopmans, whose constructive observations regarding properties of the simplex led directly to a proof of the method in the early fall of 1947". In preparing this paper, I asked Leonid Hurwicz for his recollections as to how contacts were made and what was their nature [20].

He had been employed during World War II at the Institute of Meteorology of the University of Chicago in a programme teaching mathematics and other subjects to soldiers preparing to enter training to be weather forecasters. Koopmans moved to the University of Chicago as a professor of economics and a senior member of the staff of a research organization, the Cowles Commission for Research in Economics. Hurwicz and I subsequently became Research Associates of this group.

One of Hurwicz's colleagues at the Institute of Meteorology, an applied mathematician, had moved to the National Bureau of Standards, where he became aware of the linear programming formulations of activity analysis by Dantzig and Marshall Wood (later reprinted as [53,15]). Hurwicz's colleague conveyed to him Dantzig's search for an effective algorithm. Koopmans had already told Hurwicz something of his work on the transportation problem. As Hurwicz put it to me, he didn't fully understand either linear programming or the transportation problem, but he could recognize that the two were similar.

Koopmans recognized both the conceptual importance of linear programming and the practical significance of the simplex method as making the method effective in use. He organized a conference in June, 1949, to bring together the different intellectual communities involved in the development and applications of linear programming, including its implications for economic theory; the proceedings were published [24]. The centerpiece was, of course, Dantzig's paper presenting the simplex method and published for the first time [14]. This gave an important degree of publicity to Dantzig's work and certainly helped in its diffusion. It also made clear that the mathematical concept of, "dual variables", so important in linear programming theory, were essentially what the economist called, "prices", or, more properly, "efficiency prices". This way of thinking made it easier for economists to understand the uses of linear programming and also gave the linear programmers more intuition about what their algorithm was doing.

A number of the papers in [24] applied linear programming concepts to economic theory. Most notable was [25] which reconsidered the whole theory of production, as sketched in Section 2, and showed that it could be expressed in terms of the activity analysis underlying linear programming.

There was a troubling postscript. In 1975, the Nobel Memorial Prize in Economic Sciences was awarded jointly to Koopmans and Leonid V. Kantorovich, a Soviet mathematician, "for their contributions to the theory of optimum allocation of resources", i.e. essentially for their development of linear programming.[1] Koopmans was very disturbed that Dantzig did not share the prize. He even considered rejecting the prize on this basis (he spoke to me about this possibility), but eventually did accept it. However, possibly because of these reservations, he donated one third of his share of the prize money to the International Institute for Applied Systems Analysis (with which both he and Dantzig had connections) for a lecture series.

5. The use of linear programming in economic planning

I will not be able to give a genuine survey of the applications of linear programming in economics; they have been too numerous and too diverse. I will concentrate on the works of two scholars, Hollis B. Chenery, and Alan S. Manne.

As we shall see, most of the work has been concerned with national economic planning, with particular reference to developing countries. This work has gradually dwindled over time, metamorphosing into general equilibrium models. The area of general economic analysis in which linear programming has retained and even increased its usefulness, although with more input from equilibrium analysis, has been the interaction of the energy sector with the general economy.

Chenery, who had an undergraduate engineering background and therefore some knowledge of mathematics, had worked with Leontief as a Harvard graduate student. Even before joining the Economics Department at Stanford University, he had been an official in the United States foreign aid activities and had become involved in particular with the economic development of southern Italy, a very poor region at that time (1951–1952). At Stanford, he started a research programme on economic development. His work had relied heavily on Leontief's input–output analysis, i.e. fixed coefficients for each product. I suggested to him once that economic development, if it meant rising productivity, must involve replacement of one activity by another to take advantage of changing factor availabilities, especially an increase in capital relative to labour. When he asked how this could be done, I told him of linear programming, which was already being taught at Stanford in the Statistics Department. He was interested only in results, not in theory, and saw immediately the possible applications. On my recommendation he hired a graduate student, Harvey Wagner, subsequently a major scholar in linear programming and other branches of operations research, and within a year or two was employing the simplex method in economic development models.

[1] Kantorovich had indeed formulated the linear programming model and proved the duality theorems as early as 1939 [21]. He did not, however, have any general algorithm, so he could solve only very small problems (by trial and error). There was a general (possibly inaccurate) perception at the time that the Nobel selection committee was making every effort to show its impartiality by giving the award to a Soviet contributor, but, because of political considerations, serious economic analysis in the Soviet Union was impossible. Kantorovich's work was clearly important and worth the prize. In fact, the Soviet authorities stopped his work because, although couched in terms useful to individual industries, it clearly had important resemblances to "bourgeois" economics. As a result, he could not publish a serious extended account of his work until 1959, translated into English in 1965 [22]. His work was therefore unknown to Dantzig and the entire Western linear programming community at a crucial time in the development of the subject.

Within a few years, Chenery had become not only a user of linear programming methods in economic development but a strong advocate and disseminator. With a collaborator, he published a textbook for economists [7]. Also, Chenery developed strong research organizations in economic development at Stanford and later at Harvard. He also served as a high official in the United States Agency for International Development and later as Chief Economist of the International Bank for Reconstruction and Development (the World Bank). In all these capacities, he used his position and funds to encourage the development by others, particularly junior scholars, of quantitative models, largely or entirely based on linear programming, to study the policies needed by developing countries.

Among his own applications of linear programming, three stand out in my judgment. In [5], he used linear programming to examine the interdependence of investment decisions. The model was dynamic; investment gave rise to durable capital. If one industry needs to buy some inputs from another, there must be investment in both industries. This paper showed how one could concretely compute these interdependent investments. [6] was one of the first economy-wide models, in this case a model of Israel. Whereas most previous studies had considered relatively closed economies, such as India, a small nation like Israel necessarily had a large foreign trade component. These factors were brought out very clearly.

Finally, and perhaps most influential of the three papers, [8] took further the emphasis on the foreign trade sector. Since it was assumed that foreign exchange rates as well as domestic interest rates were rigid, there emerged a "two-gap" picture, with both a domestic imbalance between saving and the need for investment and an imbalance in the balance of trade. This analysis was used both to argue for the need for foreign aid and to direct it between alternative channels. This paper had a considerable influence on research and also on policy.

Nevertheless, it proved to be misleading. In the next decade, there emerged a greater emphasis on the flexibility of the market, so that the rigidities assumed by Chenery and Strout were not as binding as they assumed. Their data were largely drawn from the Korean experience, and it was precisely Korea's success in exporting that showed the limits of the two-gap model.

Alan Manne was also a quantitative analyst from his graduate student days, also at Harvard and under the influence of Leontief. His interests were unusually industry-specific, one might say operations-research-orientated, for a graduate student in economics; his dissertation was on the scheduling of petroleum refining operations. This led to a lifelong interest in the energy sector.

His early years (1952–1956) at the RAND Corporation brought him into contact with George Dantzig and the value of linear programming. Much of his work at this tjme dealt with developments of linear programming to extend its range. One paper was more like a standard economic analysis, a linear programming model of the United States petroleum refining industry [34].

After entering academia (he has been on the faculties of Yale, Stanford and Harvard Universities), Manne constructed some of the most sophisticated linear programming models for economic development, in particular, for India [41] and for Mexico [36]. These models were more explicitly dynamic than earlier ones and brought out and resolved some of the difficulties in such models.

The model for India led to an illuminating controversy showing the interactions of policy research and politics. The research organization responsible for it was the Center for International Studies at the Massachusetts Institute of Technology. Classified work for the Department of Defense was also carried out there. In an early run of the model, a strong preference for increased support of agriculture in India was indicated. This was contrary to a prevailing view, particularly in leftist circles in India, that the cure for backwardness was rapid industrialization. As a result, there was a campaign in India attacking the model's conclusions as an American conspiracy to prevent India's growth, carried out through a research centre which was allegedly secretly under the control of the American military. I may add that no serious economist doubted the correctness of the model's emphasis on agriculture, and indeed India's later policies, after the Green Revolution, agreed.

Manne's later research exemplified and partly stimulated the general redirection of linear programming models in economics toward the interactions between a specific industry, usually the energy industry or some part of it, and the economy as a whole. The fact of their mutual interaction (the price of energy affects the economy, while the state of the economy determines in part the demand for energy) is an essential implication of economic analysis. At the same time, there is a wealth of technological information about the energy sector which can be incorporated in the model.

Manne introduced a number of new questions for study in his work on energy–economy interactions. Manne [35] considered water projects with multiple outputs (flood control, hydroelectric power, and irrigation). Manne [37] developed a dynamic model where there is an anticipated technological change; this was a powerfully influential study which introduced a number of new concepts.

These papers put the energy sector in the foreground. His subsequent studies, beginning with [38] were models which gave equal emphasis to the economy as a whole and to the energy sector. As they developed, the need for two modifications became apparent: (1) objectives and some constraints were nonlinear; and (b) the appropriate models were equilibrium models rather than simple maximization.

6. Planning models and information transfer

There was considerable interest in economic planning in postwar Europe. Of course, the Soviet Union and its satellites were in principle engaged in very comprehensive planning, though the actual procedures were very simple and far from optimal. The countries of Western Europe were also committed in varying measures to detailed planning of the economy. Linear programming appeared to offer a practical method for handling the very large-scale models implicit in planning. A conference held by the International Economic Association surveyed these developments [33]. The motives for planning which led to the employment of these models has been discussed in Section 4 above.

A critical problem in highly detailed models was that of information on the coefficients of the activities. The idea of collecting all the information in advance to set up the model was clearly preposterous; indeed that had been one of the main objections to the viability of a socialist economy by Von Mises [44,45]. The reply by proponents of socialism or any highly planned economy was to acknowledge the problem and propose an iterative approach in which some signals (say, tentative prices) would be announced and the individual firms who possessed the knowledge would choose a production plan optimal at those prices (e.g. [28]).

How could this problem of information flow be handled in a linear programming set-up? In fact, George Dantzig recognized this issue very early; when the (known) zeros in the coefficient matrix had a particular pattern (corresponding to the decomposition of production into small clusters which give rise to firms), then Dantzig and Wolfe [17] showed how the simplex method could take advantage of this structure. Issues of this kind were also stressed by Kornai and Lipták [27], who were trying to rationalize the planning process in Hungary, and by Malinvaud [31,32]. (Kornai and Lipták found that with the limited computational facilities available to them in Hungary, they could not implement the simplex method and so used a variation of the method of fictitious play.)

In the end, none of these proposals found any significant application in national economic planning on either side of the Iron Curtain.

7. Equilibrium models

Linear programming models of the economy presupposed a single decision maker with a linear objective function. The limitations of these assumptions became apparent. The nonlinearity of the objective function meant only that the emerging algorithms for nonlinear programming had to be used. Dantzig himself sought to derive a nonlinear objective function for the United States economy from the utility functions of consumers [16]; however, this study had no impact.

More serious was the presence of multiple decision makers. This gave rise to the need for equilibrium models. Indeed, this was precisely the situation envisaged in the competitive equilibrium models discussed in Section 2. The individual firms and consumers were independent decision-makers, and, in an international context, so were other governments. Each decision maker took as given the decisions of other decision makers. In most models, this interaction took the form of taking prices as given. The predictions could be made on the basis of alternative assumptions about international agreements (e.g. trade liberalization or tradable emission permits for CO_2) or on the basis of alternative policies for one single government anticipating the reactions of others.

This shift in emphasis also fitted in with a general disillusionment with economic planning. While individual government and international systems could be evaluated with these models, comprehensive planning was no longer considered either feasible or desirable.

Most of the economic literature is concerned with competitive equilibrium, in which prices play an important role, but the Nash equilibrium of games has very much the same mathematical features.

Solving equilibrium models is equivalent to finding the fixed point of a suitable mapping and is intrinsically more difficult than finding an optimum of any kind, certainly more difficult than solving a linear program. Nevertheless, linear programming had something to contribute. Lemke and Howson [29] developed an algorithm, suggested by linear programming but not identical with it, for finding the Nash equilibrium for two-person non-zero-sum games. Cottle and Dantzig [11,12] formulated a more general "linear complementary problem" and found alternative algorithms. Manne and Dantzig [39] found an application for the complementarity algorithms in the context of a dynamic model under special assumptions. Scarf [48], building on Lemke and Howson, found a general method for approximating fixed points which could be applied to solving for the solutions of competitive equilibrium models. Of course, the algorithm required much more computer time than linear programming problems of a corresponding size.

Although general equilibrium models had been solved by various *ad hoc* methods even before the publication of these papers, they gave a tremendous impetus to further developments. Many new solution methods were found, and it would go too far afield to discuss them here. I must call attention to the paper of Chao, Manne, and Wilson [3], which proposed a solution to equilibrium models by a series of linear programs. Convergence was not guaranteed, but in practice it worked very well.

The applications were even more numerous. I mention only the representative and important MERGE model, which treated world energy–economy interactions by considering separate regions of the world (with differing technologies) as participants in the world economy [40]. It built on the earlier linear programming models but went to a full equilibrium framework. However, as in many general equilibrium models today, large parts of industry retain a linear structure and use the simplex algorithm as an auxiliary to the full solution.

8. Postscript

The linear programming formulation of economy-wide problems had its impact because it caught essential properties of the system and because it could be implemented through the simplex method of George Dantzig. It was very useful in many contexts, especially in transportation and energy studies. It reached its limits, as both inner logic and the shifting evaluation of the market brought out. The algorithms used to solve the equilibrium models now dominant and much used are descended from the simplex method. No doubt there will be newer models which will replace these while incorporating much of their thinking and their modes of solution.

References

[1] K.J. Arrow, G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica 22 (1954) 265–290.
[2] E. Barone, The ministry of production in a socialist state, in: [18], 1935, pp. 245–290.
[3] H. Chao, A.S. Manne, R. Wilson, Computation of competitive equilibria by a sequence of linear programs, Econometrica 48 (1980) 1595–1615.
[4] H.B. Chenery, Engineering production functions, Quarterly Journal of Economics 63 (1949) 507–531.
[5] H.B. Chenery, The interdependence of investment decisions, in: M. Abramovitz, et al. (Eds.), The Allocation of Economic Resouces, Stanford University Press, Stanford, CA, 1959.
[6] H.B. Chenery, M. Bruno, Development alternatives in an open economy: The case of Israel, Economic Journal 72 (1962) 79–103.
[7] H.B. Chenery, P. Clark, Interindustry Economics, Wiley, New York, 1959.
[8] H.B. Chenery, A.M. Strout, Foreign assistance and economic development, American Economic Review 56 (1966) 679–733.
[9] J.B. Clark, The Distribution of Wealth, Macmillan, New York, 1899.
[10] C.W. Cobb, P.H. Douglas, A theory of production, American Economic Review 18 (1928) 139–165. Papers and Proceedings.
[11] R.W. Cottle, G.B. Dantzig, Complementary pivot theory of mathematical programming, Linear Algebra and its Applications 1 (1968) 103–125.
[12] R.W. Cottle, G.B. Dantzig, A generalization of the linear complementarity problem, Journal of Combinatorial Theory 8 (1970) 79–90.
[13] Cournot, Recherches sur les Principes Mathématiques de la Théorie des Richesses, Hachette, Paris, 1838.
[14] G.B. Dantzig, Maximization of a linear function of variables subject to linear inequalities, in: [24], 1951, pp. 339–347 (Chapter XXI).
[15] G.B. Dantzig, The programming of interdependent activities: Mathematical model, in: [24], 1951, pp. 19–32 (Chapter II).
[16] G.B. Dantzig, P.H. MCALLister, J.C. Stone, Formulating an objective for an economy, Mathematical Progamming, Series B 42 (1988) 11–32.
[17] G.B. Dantzig, P. Wolfe, Decomposition principle for linear programs, Operations Research 8 (1960) 101–111.
[18] F.A. Hayek (Ed.), Collectivist Economic Planning, George Routledge, London, 1935.
[19] F.L. Hitchcock, The distribution of a product from several sources to numerous localities, Journal of Mathematics and Physics 20 (1941) 224–230.
[20] L. Hurwicz, Personal communication to the author, 2005.
[21] L.V. Kantorovich, Matematicheskie Metody Organizatsii i Planirovaniia Proizvodstkia, Leningrad University Press, Leningrad, 1939.
[22] L.V. Kantorovich, The Best Use of Resources (G. Morton, trans.), Harvard University Press, Cambridge, MA, 1965 (Original work published 1959 in Russian).
[23] T.C. Koopmans, Optimum utilization of the transportation system, Econometrica 17 (1949) 136–146.

[24] T.C. Koopmans (Ed.), Activity Analysis of Production and Allocation, Wiley, New York, 1951, Chapman & Hall, London.
[25] T.C. Koopmans, Analysis of production as an efficient combination of activities, in [24], 1951, pp. 33–97 (Chapter III).
[26] T.C. Koopmans, S. Reiter, A model of transportation, in: [24], 1951, pp. 222–259 (Chapter XIV).
[27] J. Kornai, T. Lipták, Two-level planning, Econometrica 33 (1966) 141–169.
[28] O. Lange, On the economic theory of socialism, in: F. Taylor, O. Lange, B.E. Lipincott (Eds.), On the Economic Theory of Socialism, U. of Minnesota Press, Minneapolist, 1938, pp. 57–142.
[29] C.E. Lemke, J.T. Howson Jr., Equilibrium points of bimatrix games, Journal of the Society for Industrial and Applied Mathematics 12 (1964) 413–423.
[30] W.W. Leontief, The Structure of the American Economy 1919–1939, first ed., Harvard University Press, Cambridge, MA, 1941.
[31] E. Malinvaud, Decentralized procedures for planning, in: [33], 1967.
[32] E. Malinvaud, Notes sur l'étude des procedures de planification, Canadian Journal of Economics 1 (1968) 16–36.
[33] E. Malinvaud, M.O.L. Bacharach (Eds.), Activity Analysis in the Theory of Growth and Planning, Macmillan, London, 1967.
[34] A.S. Manne, A linear programming model of the U. S. petroleum refining industry, Econometrica 26 (1958) 67–196.
[35] A.S. Manne, Product-mix alternatives: flood control, electric power, and irrigation, International Economic Review 3 (1962) 30–59.
[36] A.S. Manne, Key sectors of the Mexican economy, in: I. Adelman, E. Thorbecke (Eds.), The Theory and Design of Economic Development, Johns Hopkins University Press, Baltimore, 1962–1972 (Chapter 10).
[37] A.S. Manne, Waiting for the breeder, Review of Economic Studies 41 (1974) 47–65.
[38] A.S. Manne, ETA-MACRO: A model of energy–economy interactions, in: C.J. Hitch (Ed.), Modeling Energy-Economy Interactions, Resources for the Future, Washington, D.C., 1977.
[39] A.S. Manne, G.B. Dantzig, A complementary algorithm for an optimal capital path with invariant proportions, Journal of Economic Theory 9 (1974) 313–332.
[40] A.S. Manne, R. Mendelsohn, R. Richels, MERGE – A model for evaluating regional and global effects of GHG reduction, Energy Policy 23 (1995) 17–34.
[41] A.S. Manne, T. Weisskopf, A dynamic multisectoral model for India, in: A.P. Carter, A. Brody (Eds.), Applications of Input-Output Analysis, North-Holland, Amsterdam, 1967–1975, pp. 70–102 (Chapter 4).
[42] L. McKenzie, On equilibrium in Graham's model of world trade and other competitive systems, Econometrica 22 (1954) 147–161.
[43] J.S. Mill, in: W. Ashley (Ed.), Principles of Political Economy, Longmans, Green, London, 1909, (first ed., 1848).
[44] L. von Mises, Die Gemeinwirtschaft, Vienna, 1922 (English translation, Socialism. Jonathan Cape, London, 1936).
[45] L. von Mises, Economic calculation in the socialist commonwealth. in: [18], 1935, pp. 87–130.
[46] A.C. Pigou, The Economics of Welfare, fourth ed., Macmillan, London, 1932.
[47] D. Ricardo, Principles of political economy and taxation, in: P. Sraffa (Ed.), in: The Works and Correspondence of David Ricardo, vol. 1, Cambridge University Press, Cambridge, UK, 1951.
[48] H.E. Scarf, The approximation of fixed points of a continuous mapping, SIAM Journal of Industrial and Applied Mathematics 15 (1967) 1328–1343.
[49] K. Schlesinger, Über die Produktionsgleichungen der ökonomischen Wertlehre, Ergebnisse eines mathematischen Kolloquiums 6 (1933–1934) 10–11.
[50] A. Wald, Über einige Gleichungssysteme der mathematischen Ökonomie, Zeitschrift für Nationalökonomie 7 (1936) 1–6.
[51] L. Walras, The Theory of Social Wealth, (W. Jaffé, Trans.), Irwin, Homewood, IL, 1954.
[52] P.H. Wicksteed, An Essay on the Coordination of the Laws of Distribution, London School of Economics, London 1894 (Reprinted 1932).
[53] M.K. Wood, G.B. Dantzig, The programming of interdependent activities: General discussion, in: [24], 1951, pp. 15–18 (Chapter I).
[54] F. Zeuthen, Das Prinzip der Knappheit, technische Kombination, und ökonomische Qualität, Zeitschrift für Nationalökonomie 4 (1932) 1–24.