ON VON NEUMANN'S MINIMAX THEOREM

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1. Introduction. It was J. von Neumann [7], [8] who first proved the minimax theorem under quite general conditions. A little later, in establishing a generalization of Brouwer's fixed-point theorem, S. Kakutani [3] gave a simple proof of this result.

In the present paper, we shall give an alternative proof of the theorem. We believe the proof to be presented is of interest, in that it enables us to derive this theorem, regardless of the dimensionality of strategy spaces, directly from the classical Brouwer's theorem, without any use of further generalizations [1], [5] of Kakutani's theorem in infinite dimensional linear spaces.

It should also be noticed that the theorem to be proved can not always be derived from such types of generalized fixed point theorems. We shall make a remark concerning this point in § 6.

In this study, the author was greatly aided by many works on this subject, especially by J. Nash's work [6] regarding the use of the fixed-point theorem.

2. Preliminaries. A linear space L over the field of real numbers R is said to be topological, if a Hausdorff topology is given in it such that the mappings

$$L \times L \ni [x, y] \longrightarrow x + y \in L$$

$$R \times L \ni [\alpha, x] \longrightarrow \alpha x \in L$$

are all continuous.

We denote by $C(A)$ the convex closure of a subset A in L.

3. Statement of the theorem. Let $K(x, y)$ be a pay-off function defined on the product space of X by Y, where X and Y are convex compact sets in topological linear spaces L and M respectively, and continuous in each variable for any fixed value of the other. Throughout what follows, these conditions on

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$K(x, y)$ should be kept in mind.

The function $K(x, y)$ will be called *quasi-concave* in x, if the inequalities

$$K(x_1, y) \geq \lambda, \quad K(x_2, y) \geq \lambda$$

imply

$$K(\alpha_1 x_1 + \alpha_2 x_2, y) \geq \lambda$$

for any real numbers λ, α_1, and α_2, where

$$\alpha_1 \geq 0, \ \alpha_2 \geq 0, \ \text{and} \ \alpha_1 + \alpha_2 = 1.$$

In other words, the quasi-concavity of $K(x, y)$ in x means therefore that the complete inverse image of every half interval $[\lambda, +\infty)$ in X is convex for any fixed y.

Dually, $K(x, y)$ will be called *quasi-convex* in y if the inequalities

$$K(x, y_1) \leq \mu, \quad K(x, y_2) \leq \mu$$

imply

$$K(x, \beta_1 y_1 + \beta_2 y_2) \leq \mu$$

for any real numbers μ, β_1, and β_2, where

$$\beta_1 \geq 0, \ \beta_2 \geq 0 \ \text{and} \ \beta_1 + \beta_2 = 1.$$

We shall prove:

VON NEUMANN'S MINIMAX THEOREM: *If $K(x, y)$ is quasi-concave in x and quasi-convex in y, then*

$$\max_{x \in X} \min_{y \in Y} K(x, y) = \min_{y \in Y} \max_{x \in X} K(x, y).$$

4. Proof of the theorem. To prove the theorem, we first consider the sets

$$E_\lambda = \{x \,;\, x \in X;\, K(x, y) \geq \lambda \ \text{for any} \ y \in Y\},$$

$$F_\mu = \{y \,;\, y \in Y;\, K(x, y) \leq \mu \ \text{for any} \ x \in X\},$$

where λ and μ are arbitrary real numbers.

These sets are evidently closed (convex) subsets of X and Y respectively. Next put

$$\lambda_0 = \sup_{E_\lambda \neq \phi} \lambda ; \quad \mu_0 = \inf_{F_\mu \neq \phi} \mu ,$$

where ϕ indicates the empty set. Then it follows from the compactness of X and Y that

$$E_{\lambda_0} \neq \phi; \quad F_{\mu_0} \neq \phi,$$

$$\lambda_0 < +\infty; \quad \mu_0 > -\infty.$$

Let x^0 and y^0 be any points of E_{λ_0} and F_{μ_0}, respectively; then we have, by definition,

$$\lambda_0 \leq K(x^0, y^0) \leq \mu_0 .$$

If we succeed in showing that $\lambda_0 = \mu_0$, the definition of E_{λ_0} and F_{μ_0} yields immediately

$$K(x^0, y) \geq \lambda_0 = K(x^0, y^0) = \mu_0 \geq K(x, y^0)$$

for any $x \in X$ and any $y \in Y$; that is, $[x^0, y^0]$ is a saddle-point of $K(x, y)$. Thus we have only to see that $\lambda_0 = \mu_0$.

For this aim, let ϵ be an arbitrary positive number. Then we have

$$E_{\lambda_0 + \epsilon} = \phi; \quad F_{\mu_0 - \epsilon} = \phi.$$

The statement

$$E_{\lambda_0 + \epsilon} = \phi$$

implies the fact that for every $x \in X$ there exists some $y \in Y$ such that

(1) $$K(x, y) < \lambda_0 + \epsilon \equiv \overline{\lambda}.$$

The statement

$$F_{\mu_0 - \epsilon} = \phi$$

implies the fact that for every $y \in Y$ there exists some $x \in X$ such that

$$K(x, y) > \mu_0 - \epsilon \equiv \bar{\mu}. \tag{2}$$

Let us put

$$U_y = \{x \,;\, x \in X,\ K(x, y) < \bar{\lambda}\},$$

$$V_x = \{y \,;\, y \in Y,\ K(x, y) > \bar{\mu}\}.$$

Then U_y and V_x are open sets in X and Y, respectively. Furthermore, in view of (1) and (2), we have

$$X \subset \bigcup_{y \in Y} U_y \,;\, Y \subset \bigcup_{x \in X} V_x.$$

Then, by virtue of the compactness of X and Y, there exist two systems of finite numbers of points $\{a_i \,;\, i = 1, 2, \cdots, s\}$ and $\{b_j \,;\, j = 1, 2, \cdots, t\}$ such that

$$X \subset \bigcup_{j=1}^{t} U_{b_j} \,;\, Y \subset \bigcup_{i=1}^{s} V_{a_i} \,;$$

these relations imply

$$\min_j K(x, b_j) < \bar{\lambda} \qquad \text{for any } x \in X, \tag{3}$$

$$\max_i K(a_i, y) > \bar{\mu} \qquad \text{for any } y \in Y. \tag{4}$$

Put now

$$\phi_i(y) = \max(0, K(a_i, y) - \bar{\mu}), \tag{5}$$

$$\psi_j(x) = \max(0, \bar{\lambda} - K(x, b_j)). \tag{6}$$

Then, by definition,

$$\phi_i(y) \geq 0,\ \psi_j(x) \geq 0,$$

and, by virtue of (3) and (4),

$$\sum_1^s \phi_i(y) > 0;\ \sum_1^t \psi_j(x) > 0.$$

We consider the following mapping:

$$X \times Y \ni [x, y] \longrightarrow \left[\frac{\sum \phi_i(y) a_i}{\sum \phi_i(y)} , \frac{\sum \psi_j(x) b_j}{\sum \psi_j(x)} \right] \in X \times Y.$$

This mapping is clearly a continuous mapping from $X \times Y$ into itself.

Denote by A and B the finite point-sets $\{a_i\}$ and $\{b_j\}$ respectively. Then $C(A) \times C(B)$ is continuously mapped into itself by the above mapping. Since $C(A) \times C(B)$ is homeomorphic to a convex compact set in a Euclidean space, Brouwer's fixed-point theorem applies to it. Thus there exists a pair $[\hat{x}, \hat{y}]$ in $C(A) \times C(B) \subset X \times Y$ such that

$$\hat{x} = \frac{\sum \phi_i(\hat{y}) a_i}{\sum \phi_i(\hat{y})} ; \quad \hat{y} = \frac{\sum \psi_j(\hat{x}) b_j}{\sum \psi_j(\hat{x})} .$$

Now, by the definition of $\phi_i(y)$, we have $K(a_i, \hat{y}) > \bar{\mu}$ for i such that $\phi_i(\hat{y}) > 0$; hence the quasi-concavity of $K(x, y)$ yields

$$(7) \qquad K(\hat{x}, \hat{y}) > \bar{\mu}.$$

On the other hand, by the definition of $\psi_j(x)$, we have $K(x, b_j) < \bar{\lambda}$ for j such that $\psi_j(\hat{x}) > 0$; hence the quasi-convexity of $K(x, y)$ yields

$$(8) \qquad K(\hat{x}, \hat{y}) < \bar{\lambda}.$$

Therefore (7) and (8) imply $\bar{\mu} < \bar{\lambda}$, which means

$$(9) \qquad \mu_0 < \lambda_0 + 2\epsilon.$$

Thus the arbitrariness of ϵ implies $\mu_0 \leq \lambda_0$. Hence we have, together with the fact previously established, $\lambda_0 = \mu_0$, which was to be shown.

As a special case of this theorem, we obtain the generalized J. Ville's theorem: *Let $f(x, y)$ be a continuous real-valued function defined on the product space of a compact Hausdorff space G by a space of the same category H. Let further P and Q be the totalities of all regular Borel probability measures on G and H respectively. Then we have*

$$\max_{\xi \in P} \min_{\eta \in Q} \iint_{G \times H} f(x, y) d\xi d\eta = \min_{\eta \in Q} \max_{\xi \in P} \iint_{G \times H} f(x, y) d\xi d\eta.$$

5. Approximating finite strategy spaces. Let X, Y, and $K(x, y)$ be the same as before. Let further S and T be finite point-sets in X and Y respectively. By virtue of the theorem just proved, the game with the same payoff is determined, even if $C(S)$ and $C(T)$ are employed as strategy spaces. Denote by σ and τ the values of the original game and the game mentioned above respectively.

Then $C(S)$ and $C(T)$ will be called ϵ-approximating finite strategy spaces provided $|\sigma - \tau| \leq \epsilon$.

We shall next establish their existence for any $\epsilon > 0$. But, to establish this fact, we do not need a new discussion. Indeed, just $C(A)$ and $C(B)$ in the preceding section serve as ϵ-approximating finite strategy-spaces.

In fact, putting $\lambda_0 = \mu_0 \equiv \sigma$, we obtain, in view of (3) and (4),

$$(10) \qquad \tau = \max_{x \in C(A)} \min_{y \in C(B)} K(x, y) \leq \max_{x \in X} \min_{y \in C(B)} K(x, y) \leq \sigma + \epsilon,$$

$$(11) \qquad \tau = \min_{y \in C(B)} \max_{x \in C(A)} K(x, y) \geq \min_{y \in Y} \max_{x \in C(A)} K(x, y) \geq \sigma - \epsilon.$$

This proves that $|\sigma - \tau| \leq \epsilon$; (10) means moreover that the minimizing player can, by choosing an appropriate strategy from $C(B)$, secure for himself a gain $\geq -(\sigma + \epsilon)$, irrespective of what the opponent does within X. And (11) means that the maximizing player can, by choosing an appropriate strategy from $C(A)$, secure for himself a gain $\geq \sigma - \epsilon$, irrespective of what the opponent does within Y.

6. Miscellaneous remarks. The implication of the assumption that $K(x, y)$ is continuous in each variable for any fixed value of the other is a little delicate. The Kakutani-type approach to the minimax theorem imposes only a very weak algebraic restriction on $K(x, y)$: namely, $\Phi(x)$ of (12) as well as $\Psi(y)$ of (13) should be convex sets. But it requires a rather strong topological condition; the continuity of the payoff in the variable $[x, y]$. Our minimax theorem sacrifices, on the contrary, the algebraic conditions to a certain degree for the benefit of the weaker topological condition [1] on payoffs. This implicit result of

[1] In case of bilinear, or convex, games [2, 5, 4], the topological condition on payoffs can be further weakened. As to the algebraic conditions, it should be noticed that neither Fan's convexity concept [2] nor our quasi-convexity concept includes the other. Since Fan's convexity premises no linear spaces, this quite general concept can not be deduced from ours. On the other hand, one may point out, by means of the following example, that our quasi-convexity is not always implied by Fan's convexity: $K(x, y) = -x^2/[(x-y)^2 + 1]$ for $-1 \leq x, y \leq 1$. This payoff is not convex in y in Fan's sense. Indeed, there exists no y such that $K(x,y) \leq [K(x, 1) + K(x,-1)]/2$ for all x. But it is quasi-concave in x and quasi-convex in y.

our theorem is clarified by means of an example.

K. Fan [1] has recently established, in generalizing Kakutani's fixed-point theorem, the following result: Let L be a locally convex topological linear space and X a convex compact subset of it. For any upper semi-continuous point-set transformation, $x \longrightarrow \Phi(x)$, where $\Phi(x)$ is a closed convex subset of X for each $x \in X$, there exists a point x such that $x \in \Phi(x)$. Here, by upper semi-continuity we understand: for any x and any neighborhood U of the null point there exists a neighborhood V of x such that

$$\Phi(V) \subset \Phi(x) + U.$$

If, in our theorem, $K(x, y)$ is continuous with respect to the variable $[x, y] \in X \times Y$, it is easily seen that the point-to-set transformations

$$(12) \qquad X \ni x \longrightarrow \Phi(x) = \{y \,;\, y \in Y, \min_y K(x, y) = K(x, y)\},$$

$$(13) \qquad Y \ni y \longrightarrow \Psi(y) = \{x \,;\, x \in X, \max_x K(x, y) = K(x, y)\},$$

are all upper semi-continuous. Thus any fixed point of the transformation,

$$X \times Y \ni [x, y] \longrightarrow \Psi(y) \times \Phi(x)$$

is a desired saddle-point of $K(x, y)$. However, our theorem can not always be obtained by this method, even if the strategy-spaces are embedded in locally convex spaces. We give here a simple example of $K(x, y)$ for which the transformations (12), (13) are not upper semi-continuous, which however meets the conditions of our theorem.

Let, indeed, (x, y) be the inner product of an infinite dimensional separable real Hilbert space L. Denote by S the unit sphere of L. Then the game with

$$K(x, y) \equiv (x, y), \quad X = Y \equiv S$$

meets the conditions of our theorem, if viewed from the standpoint of the weak topology. (Moreover, it is almost obvious that this game has only one saddle-point: [0, 0].) But, for this payoff, the transformations (12) and (13) are not upper semi-continuous. We shall show, for instance, that there exist two sequences $\{x_n\}$, $\{y_n\}$ in S, where

$$x_n \longrightarrow x, \quad y_n \longrightarrow y$$

weakly and $x_n \in \Psi(y_n)$, but $x \notin \Psi(y)$.

Now, since

$$||y|| = \max_{x \in S} (x, y),$$

it follows that

$$\Psi(y) = \{x\, ;\, ||x|| \leq 1,\, (x, y) = ||y||\}.$$

Let $\{z_n\}$ be a complete orthonormal system. Next, let us put

$$x_n = \sqrt{2}\, y_n\, ;\;\; y_n = z_1/2 + z_{n+1}/2\, ;\;\; x = \sqrt{2}\, y,\;\; y = z_1/2.$$

Then it is obvious that

$$||x_n|| = 1,\;\; ||y_n|| = 1/\sqrt{2},\;\; \text{and}\;\; x_n \longrightarrow x,\;\; y_n \longrightarrow y$$

weakly in S. Furthermore, $x_n \in \Psi(y_n)$, because $(x_n, y_n) = ||y_n||$. But $x \notin \Psi(y)$, since $(x, y) \neq ||y||$.

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*The author, after having finished the present note, read [2] and [4]; these papers were recommended to him, the former by Dr. M. Dresher and the latter by Professor J. von Neumann.

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