A GENERALIZATION OF BROUWER'S FIXED POINT THEOREM

By Shizuo Kakutani

The purpose of the present paper is to give a generalization of Brouwer's fixed point theorem (see [1][1]), and to show that this generalized theorem implies the theorems of J. von Neumann ([2], [3]) obtained by him in connection with the theory of games and mathematical economics.

1. The fixed point theorem of Brouwer reads as follows: *if* $x \to \varphi(x)$ *is a continuous point-to-point mapping of an r-dimensional closed simplex S into itself, then there exists an* $x_0 \in S$ *such that* $x_0 = \varphi(x_0)$.

This theorem can be generalized in the following way: Let $\mathfrak{R}(S)$ be the family of all closed convex subsets of S. A point-to-set mapping $x \to \Phi(x) \in \mathfrak{R}(S)$ of S into $\mathfrak{R}(S)$ is called upper semi-continuous if $x_n \to x_0$, $y_n \in \Phi(x_n)$ and $y_n \to y_0$ imply $y_0 \in \Phi(x_0)$. It is easy to see that this condition is equivalent to saying that the graph of $\Phi(x)$: $\sum_{x \in S} x \times \Phi(x)$ is a closed subset of $S \times S$, where $\times$ denotes a Cartesian product. Then the generalized fixed point theorem may be stated as follows:

Theorem 1. *If* $x \to \Phi(x)$ *is an upper semi-continuous point-to-set mapping of an r-dimensional closed simplex S into* $\mathfrak{R}(S)$, *then there exists an* $x_0 \in S$ *such that* $x_0 \in \Phi(x_0)$.

Proof. Let $S^{(n)}$ be the n-th barycentric simplicial subdivision of S. For each vertex x^n of $S^{(n)}$ take an arbitrary point y^n from $\Phi(x^n)$. Then the mapping $x^n \to y^n$ thus defined on all vertices of $S^{(n)}$ will define, if it is extended linearly inside each simplex of $S^{(n)}$, a continuous point-to-point mapping $x \to \varphi_n(x)$ of S into itself. Consequently, by Brouwer's fixed point theorem, there exists an $x_n \in S$ such that $x_n = \varphi_n(x_n)$. If we now take a subsequence $\{x_{n_\nu}\}$ $(\nu = 1, 2, \cdots)$ of $\{x_n\}$ $(n = 1, 2, \cdots)$ which converges to a point $x_0 \in S$, then this x_0 is a required point.

In order to prove this, let Δ_n be an r-dimensional simplex of $S^{(n)}$ which contains the point x_n. (If x_n lies on the lower-dimensional simplex of $S^{(n)}$, then Δ_n is not uniquely determined. In this case, let Δ_n be any one of these simplexes.) Let $x_0^n, x_1^n, \cdots, x_r^n$ be the vertices of Δ_n. Then it is clear that the sequence $\{x_i^{n_\nu}\}$ $(\nu = 1, 2, \cdots)$ converges to x_0 for $i = 0, 1, \cdots, r$, and we have $x_n = \sum_{i=0}^{r} \lambda_i^n x_i^n$ for suitable $\{\lambda_i^n\}$ $(i = 0, 1, \cdots, r; n = 1, 2, \cdots)$ with $\lambda_i^n \geqq 0$ and $\sum_{i=0}^{r} \lambda_i^n = 1$. Let us further put $y_i^n = \varphi_n(x_i^n)$ $(i = 0, 1, \cdots, r;$

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[1] Numbers in brackets refer to the list of references at the end.

$n = 1, 2, \cdots$). Then we have $y_i^n \in \Phi(x_i^n)$ and $x_n = \varphi_n(x_n) = \sum_{i=0}^{r} \lambda_i^n y_i^n$ for $n = 1, 2, \cdots$. Let us now take a further subsequence $\{n'_\nu\}$ ($\nu = 1, 2, \cdots$) of $\{n_\nu\}$ ($\nu = 1, 2, \cdots$) such that $\{y_i^{n'_\nu}\}$ and $\{\lambda_i^{n'_\nu}\}$ ($\nu = 1, 2, \cdots$) converge for $i = 0, 1, \cdots, r$, and let us put $\lim_{\nu\to\infty} y_i^{n'_\nu} = y_i^0$ and $\lim_{\nu\to\infty} \lambda_i^{n'_\nu} = \lambda_i^0$ for $i = 0, 1, \cdots, r$. Then we have clearly $\lambda_i^0 \geqq 0$, $\sum_{i=0}^{n} \lambda_i^0 = 1$ and $x_0 = \sum_{i=0}^{r} \lambda_i^0 y_i^0$. Since $x_i^{n'_\nu} \to x_0$, $y_i^{n'_\nu} \in \Phi(x_i^{n'_\nu})$ and $y_i^{n'_\nu} \to y_i^0$ for $i = 0, 1, \cdots, r$, we must have, by the upper semi-continuity of $\Phi(x)$, $y_i^0 \in \Phi(x_0)$ for $i = 0, 1, \cdots, r$, and this implies, by the convexity of $\Phi(x_0)$, that $x_0 = \sum_{i=0}^{r} \lambda_i^0 y_i^0 \in \Phi(x_0)$. Thus the proof of Theorem 1 is completed.

Remark. It is easy to see that Brouwer's fixed point theorem is a special case of Theorem 1 when each $\Phi(x)$ consists only of one point $\varphi(x)$. In this case, the upper semi-continuity of $\Phi(x)$ is nothing but the continuity of $\varphi(x)$.

As an immediate consequence of Theorem 1 we have

COROLLARY. *Theorem 1 is also valid even if S is an arbitrary bounded closed convex set in a Euclidean space.*

Proof. Take a closed simplex S' which contains S as a subset, and consider a continuous retracting point-to-point mapping $x \to \psi(x)$ of S' onto S. ($\psi(x) = x$ for any $x \in S$ and $\psi(x) \in S$ for any $x \in S'$.) Then $x \to \Phi(\psi(x))$ is clearly an upper semi-continuous point-to-set mapping of S' into $\mathfrak{R}(S) \subseteq \mathfrak{R}(S')$. Hence, by Theorem 1, there exists an $x_0 \in S'$ such that $x_0 \in \Phi(\psi(x_0))$. Since $\Phi(\psi(x_0)) \subseteq S$, we must have $x_0 \in S$ and consequently, by the retracting property of $\psi(x)$, $x_0 \in \Phi(x_0) \subseteq S$. This completes the proof of the corollary.

2. THEOREM 2. *Let K and L be two bounded closed convex sets in the Euclidean spaces R^m and R^n respectively, and let us consider their Cartesian product $K \times L$ in R^{m+n}. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of all $y \in L$ such that $(x_0, y) \in U$, is non-empty, closed and convex, and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is non-empty, closed and convex. Under these assumptions, U and V have a common point.*

Proof. Put $S = K \times L$, and let us define a point-to-set mapping $z \to \Phi(z)$ of S into $\mathfrak{R}(S)$ as follows: $\Phi(z) = V_y \times U_x$ if $z = (x, y)$. Since U and V are both closed by assumption, $\Phi(z)$ is clearly upper semi-continuous. Hence, by the corollary of Theorem 1, there exists a point $z_0 \in K \times L$ such that $z_0 \in \Phi(z_0)$. In other words, there exists a pair of points x_0 and y_0, $x_0 \in K$, $y_0 \in L$ such that $(x_0, y_0) \in V_{y_0} \times U_{x_0}$ or equivalently, $x_0 \in V_{y_0}$ and $y_0 \in U_{x_0}$. This means that $z_0 = (x_0, y_0) \in U \cdot V$, and the proof of Theorem 2 is completed.

Remark. Theorem 2 is due to J. von Neumann [3], who proved this by using a notion of integral in Euclidean spaces. The proof given above is simpler.

This theorem has applications to the problems of mathematical economics as was shown by J. von Neumann.

THEOREM 3. *Let $f(x, y)$ be a continuous real-valued function defined for $x \epsilon K$ and $y \epsilon L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces R^m and R^n. If for every $x_0 \epsilon K$ and for every real number α, the set of all $y \epsilon L$ such that $f(x_0, y) \leqq \alpha$ is convex, and if for every $y_0 \epsilon L$ and for every real number β, the set of all $x \epsilon K$ such that $f(x, y_0) \geqq \beta$ is convex, then we have*

$$\max_{x \epsilon K} \min_{y \epsilon L} f(x, y) = \min_{y \epsilon L} \max_{x \epsilon K} f(x, y).$$

Proof. Let U and V be the sets of all $z_0 = (x_0, y_0) \epsilon K \times L$ such that $f(x_0, y_0) = \min_{y \epsilon L} f(x_0, y)$ and $f(x_0, y_0) = \max_{x \epsilon K} f(x, y_0)$ respectively. Then it is easy to see that both U and V satisfy the conditions of Theorem 2. Hence, by Theorem 2, there exists a point $z_0 = (x_0, y_0) \epsilon K \times L$ such that $z_0 \epsilon U \cdot V$ or equivalently, $f(x_0, y_0) = \min_{y \epsilon L} f(x_0, y) = \max_{x \epsilon K} f(x, y_0)$. Consequently, we have $\min_{y \epsilon L} \max_{x \epsilon K} f(x, y) \leqq \max_{x \epsilon K} f(x, y_0) = f(x_0, y_0) = \min_{y \epsilon L} f(x_0, y) \leqq \max_{x \epsilon K} \min_{y \epsilon L} f(x, y)$. Since it is clear that we have $\min_{y \epsilon L} \max_{x \epsilon K} f(x, y) \geqq \max_{x \epsilon K} \min_{y \epsilon L} f(x, y)$, the proof of Theorem 3 is completed.

Remark. Theorem 3 is one of the fundamental theorems in the theory of games developed by J. von Neumann [2].

In concluding this paper I should like to express my hearty thanks to Dr. A. D. Wallace for his kind discussions on this problem. He has also obtained analogous results for trees. (A. D. Wallace [4].)

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