A Modern Theorist's Vindication of Adam Smith

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Inside every classical economist is a modern economist trying to get out. In rereading the *Wealth of Nations*, it seems to me that with a little midwifery sleight of hand, one can extract from Adam Smith a valuable model that vindicates him from criticisms of Ricardo and Marx and from the general supercilious discounting of Smith as an unoriginal theorist who is logically fuzzy and eclectically empty. My general finding, as reported in these brief literary words here today and in a companion mathematical appendix, provides a vindication of Adam Smith and serves, in my mind at least, to raise his stature as an economic theorist, both absolutely and in comparison with his predecessors and successors.

I. Views on Smith

Smith is admired for his eclectic wisdom about developing capitalism, and for his ideological defense of competitive *laissez faire* as against blundering Mercantilist interferences with the market. His analysis of the division of labor, like Allyn Young's analysis of increasing returns in the 1920's, is thought to be seminal for the understanding of change, for the Chamberlinian deviations from perfect competition, and for the young Marx's concept of *alienation* of the overspecialized worker.

But there you have it. As a pure theorist, Adam Smith is written down precisely because of his fuzzy eclecticism. His natural prices and wages are thought to be merely the resultants of long-run supply and demand. His pluralistic decomposition of price and of Net National Product *(NNP)* into components of wages, land rent, and of profit is criticized as emptily tautological. After his good start with the labor theory of value, Smith is thought to have blotted his copybook by introducing *ad hoc* and not-fully-explained deductions from labor's full share by landowners and capitalist owners of stock. Even Smith's accounting decomposition of national income into value added elements of wages, rents and profits has been attacked in *Capital*, Volume 2, as involving vicious-circle reasoning. Too often theorists contrast Adam Smith to his disfavor with his brilliant predecessor, David Hume, and brilliant successor, David Ricardo.

II. The Case for Smith

My reading is otherwise.

1) Smith's value-added accounting is shown to be correct by Leontief-Sraffa modeling.

2) His pluralistic supply-and-demand analysis in terms of all three components of wages, rents, and profits is a valid and valuable anticipation of general equilibrium modeling.

3) His vision of transient growth from invention and capital accumulation, which is brought to an equilibrium end with a low rate of profit and a high total of land rent, is *isomorphic* with the model of Ricardo, Malthus, and Marx. But Smith is less guilty than these three of believing in a rigid subsistence-wage supply of labor in the short and intermediate run; so Smith's transient rise in wage rates is a credit to his model's realism, wherever it deviates in emphasis from its successors.

As a theorist, I do find things to criticize in Smith. Thus, he seems never to have known how to put net capital formation into his Net National Product concept. His exposition is 1776, not 1876 or 1976, in its vagueness. However, with careful reading, we do infer in the *Wealth of Nations* a complete and valuable theoretical model.

Finally, I omit in this brief paper discussion of

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pseudo-problems that have monopolized the Smith-Ricardo literature.

Although my axioms are those of the 1776 Adam Smith, my analysis from them utilizes 1976 mathematical methods, including convenient duality theory. Today, heavy mathematics will be eschewed and reference merely made to the accompanying mathematical appendix.

III. Smith's Assumptions

i) Goods, e.g., food and clothing, are produced in a time-phased way out of land and "doses" of labor-cum-raw-materials.

ii) To arrive at net consumable outputs of goods, e.g., food and clothing, one must subtract from the gross production of each the amounts of that respective good used as input components of the various industry doses.

iii) A ration of subsistence goods per laborer, e.g., m_1 of food and m_2 of clothing, is required to produce and reproduce the population. When the worker's money wage can buy more than the subsistence vector, population grows at a positive percentage rate; when the money wage buys less than subsistence, population declines exponentially; at the subsistence wage, population is constant.

iv) Workers never save and invest. Owners of land and of raw material inputs spend their wealth on food and clothing as they will. So long as the profit rate is above some minimal subsistence rate for saving, which might be zero after allowing for stochastic losses and management expenses, nonworkers do positive saving, which is never aborted. Below that minimal profit rate, nonlaborers decumulate or dissave; at the minimal profit rate, net saving and net accumulation is zero.

v) Perfect competition prevails. Land use is auctioned off for rentals. Free entry and constant returns to scale prevail. Knowledge is, or soon becomes, general.

IV. Smith's Implications

A logician, turning his deductive crank, would deduce the following properties of Smith's system.

1) Suppose it begins in long-run equilibrium. Wages are at the subsistence level. The profit rate is minimal. Depending on the pattern of nonlaborer tastes for food and clothing, land rent will be high or low; land-intensive food price will be high or low relative to clothing price; the size of the population and of the various components of raw material inventories will be high or low depending on nonlaborer tastes; and so will depend the relative distribution of *NNP* between land owners' rent and workers' wages, to say nothing of capitalists' profits if the minimal interest rate is not zero.

Most of this Ricardo missed. Some Malthus caught. Smith denies none of this, but offers little in detail.

2) Now let there be an invention. It will be viable only if, in some industry, it raises one or more of the following: the real wage there, the real rent there, or the profit rate. Except for the singular case where its incidence happens to be solely to raise land rent everywhere, the invention must transiently raise one, or more probably both, of the profit rate and the real wage rate. This initiates population growth and capital accumulation. We are in Smith's "cheerful" transient state of growth—like England rather than China or India. But ultimately, as in China and Holland, the land fills up; the law of diminishing returns on fixed land operates.

3) The system relapses into Smith's "dull state" of equilibrium with subsistence profit rate, subsistence real wage rate, and *enhanced* land rent. In effect, Smith's system maximizes rent!

4) If inventions keep recurring, the system goes through a Brownian motion in which profit rates and real wage rates average out *above* their subsistence levels, perhaps being trendless.

5) The model captures the general behavior of economic history these last two centuries if only Smith modifies his demographic hypothesis that population explodes whenever the real wage is above an unchanged subsistence level. If the needed ration of subsistence itself grows exponentially in time, then the presumption is that (a) the real wage will oscillate around an upward-rising exponential

trend, with the labor force possibly growing slowly; (b) the profit rate will meander, averaging out positive and inducing growing capital inventories; (c) land rent will tend to rise, subject to any land-saving biases in invention and to the subtraction from its rise due to the rise in real wages; (d) once we allow for alternative ways of producing the same things and for any biases in inventions, relative wage and nonwage shares of *NNP* cannot be predicted to show any definite trends; but that does not mean that minor changes contrived in labor supply can necessarily much alter the relative wage share.

These last few propositions sound much like what Simon Kuznets reports for the laws of motion of western economies, even if Ricardo and Marx failed to come as close to them as did the *Wealth of Nations*. Hats off, I say, to Adam Smith.

6) If we add to the above model a declining supply of primary "land"—that is, declining stocks of nonreproducible natural resources, such as rich seams of metal ores and coal and exhausted geologic deposits of oil and gas—we are prepared for the Club of Rome's future.

It becomes a race between invention (spontaneous and induced) and dwindling natural resources per head: the profit rate can be expected to meander in no predictable way, the real wage to grow at a slower rate (or even to suffer a declining trend). Nonwage and nonprofit share, always so important in explaining the great historic fortunes, may possibly rise. Analysis can carry prophecy no further.

V. Verdict

It is serendipitous to be able to announce, not the Scottish verdict *unproven*, but the happy finding that Adam Smith comes through with flying colors from a modern postmortem, provided we conduct it with the modicum of charity due an early pioneer.

Mathematical Appendix

The following equations vindicate Adam Smith from the principal indictments against him, and also reveal the half-untruth present in his INVISIBLE HAND doctrine.

Productivity Assumptions

Smith assumes that any of commodities, $(q_1, \ldots, q_n)$, is produced by its industry out of its labor inputs, $(L_1, \ldots, L_n)$, its land inputs $(T_1, \ldots, T_n)$, and out of produced inputs such as raw materials (or durable equipments) purchased by the various industries: so q_j will require for its production, along with T_j and L_j, also $(q_{1j}, \ldots, q_{nj})$. Smith's production functions embodying known technology can be written as

$$(1)\quad q_j(t+1) = F_j[T_j(t), L_j(t), q_{1j}(t), \ldots, q_{nj}(t)] \qquad (j = 1, \ldots, n)$$

Note the time-phasing of production in (1): inputs are needed prior to the appearance of output. In (1), T_j could be a vector of elements representing heterogeneous lands of different grades.

To arrive at net available *consumption* amounts of the ith goods, $[C_i(t)]$, one writes:

$$(2)\quad C_i(t) = q_i(t) - \sum_{j=1}^{n} q_{ij}(t) \geq 0, \qquad q_{ij}(t) \geq 0$$

Whereas a modern neoclassical economist might wish to assume that inputs can be substituted for each other in a smooth way so that $F_j[\ ]$ all have well-defined partial derivatives, a classical economist like Smith usually thought that a variable "dose" of labor-cum-raw-materials could be applied to fixed land more intensively or less intensively. So one rewrites (1) as

$$(3)\quad q_j(t+1) = F_j[T_j(t), V_j(t)]$$

$$(4)\quad V_j(t) = \text{Min}\,[L_j(t)/a_{0j}, q_{1j}(t)/a_{1j}, \ldots q_{nj}(t)/a_{nj}]$$

The a_{ij}'s are non-negative. When some a is zero, it is as if its argument is absent from the expression Min[ ].

The production functions in equation (3) are postulated to have simple properties once the scale of production goes beyond the initial levels at which the division of labor does not pay. Each $F_j[\ ]$ is concave, homogeneous-first-degree, and differentiable:

$$(5)\quad F_j[\lambda T,\ \lambda V] \equiv \lambda F_j[T, V]$$

$$F_j[T+\Delta T,\ V+\Delta V] - F_j[T,V] \geq F_j[T+2\Delta T, V+2\Delta V] - F_j[T+\Delta T, V+\Delta V]$$

$$\partial F_j[T,\ V]/\partial V > 0,\qquad F_j[T,\ V]\ - V\partial F_j[T,\ V]/\partial V \geq 0$$

Finally, Smith even before Malthus and Marx believed that human labor itself had a reproduction cost at that level of *subsistence* (food, clothing, etc.) at which a family could manage to reproduce itself by mortality survival and procreation. The long-run reproduction cost of total labor, $\Sigma_1^n L_j = L$, is defined per unit of L by the nonzero column vector of needed subsistence: m_1 of q_1, m_2 of q_2, ..., m_n of q_n:

$$(6)\qquad \mathbf{m} = [m_i] = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \geq 0$$

If the real wage exceeded the subsistence vector **m**, L_t would grow; if it fell below **m**, L_t would decline; at exactly **m**, Smith's stationary state would prevail. Evaluating the iron ration of subsistence at its market prices, $\Sigma_1^n P_j m_j$, we compare it with the market wage, W, thereby to determine the rate of population growth. Smith's simplest Malthusian relation, I write as

$$(7)\quad (L_{t+1} - L_t)/L_t = f[1 - \sum_1^n (P_j/W)m_j]$$
$$f[0] = 0, f'[\ ] > 0, f[\ ] \geq -1$$

Clearly, when the real wage is at the subsistence level **m**, population growth ceases.

Smith's Early "Rude State"

For one page, Smith does have a "labor theory of value," writing (*Wealth of Nations*, Book I, ch. 6):

> In that early and rude state of society which precedes both the accumulation of stock ["capital"] and the appropriation of [scarce] land, the proportion between the quantities of labour necessary for acquiring different objects seems to be the only circumstance which can afford any rule for exchanging them for one another... what is usually the produce of two days' or hours' labour, should be worth double of what is usually the produce of one day's or one hour's labour. . . .
>
> In this state of things, the whole produce of labour belongs to the labourer. . . .

We can make logical, even if not historical and anthropological sense of this, by postulating that land is so abundant as to be redundant and *free*, with the ratio $\Sigma_1^n T_j/\Sigma_1^n L_j$ so great as to make land ignorable. To make inventories of raw materials and crude tools ignorable takes a greater stretch of the imagination. I cut the knot by postulating that outputs and inputs are *simultaneous* rather than lagged as in equations (1) and (3).

With land redundant, so that no increase in T_j has any incremental effect on q_j output, one rewrites equations (2) and (3) in the rude state as

$$(8)\quad q_j(\mathrm{t}) = \alpha_j V_j(\mathrm{t}) = V_j(\mathrm{t}),\ (j=1,\ \ldots,\ n)$$
$$= \mathrm{Min}\ [L_j(\mathrm{t})/a_{0j}, q_{1j}(\mathrm{t})/a_{1j}, \ldots, q_{nj}(\mathrm{t})/a_{nj}]$$

Here, by proper choice of dimensional units of goods or of doses, we can suppress the $[\alpha_j]$ coefficients.

Indeed, if the rude state is in exact stationary equilibrium, we can ignore all timing designations and define that exact state by the following specializations of (1)–(8):

$$(9)\quad L - \sum_1^n L_j = 0$$

$$q_i - \sum_{j=1}^n q_{ij} - m_i L = 0, \qquad (i=1, \ldots, n)$$

By virtue of equation (3)'s definition of the fixed components of the doses, these relations become

$$(10)\quad L-\sum_{1}^{n} a_{0j}q_j=0$$

$$-m_iL+q_i-\sum_{1}^{n} a_{ij}q_j=0,\quad (i=1,\ldots n)$$

These linear equations can have a positive solution $(L, q_1, \ldots, q_n)$ only if the following technological conditions for the rude state are exactly met:

$$(11)\quad 0=\begin{vmatrix} 1 & -a_{01}\ldots & -a_{0n} \\ -m_1 & 1-a_{11}\ldots & -a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -m_n & 1-a_{n1}\ldots & \end{vmatrix}$$
$$=\det[\mathbf{I}-a_{ij}-m_ia_{0j}]$$
$$=\det[\mathbf{I}-\mathbf{a}-\mathbf{m}\mathbf{a}_0]$$

where $\mathbf{a}_0$ is the row vector of *direct labor* requirements, $[a_{0j}]$, $\mathbf{m}$ is the column vector of subsistence requirements per worker, $[m_i]$, and $\mathbf{a}$ is the n-by-n square Leontief matrix of input-output coefficients, $[a_{ij}]$.

We now vindicate Smith's equating the competitive pricing relations of his rude state with their embodied total labor requirements (direct plus indirect). There are of course no further components of the prices of the goods, $[P_1, \ldots, P_n]=[P_j]=\mathbf{P}$, than the wage component involving the money wage, W: land rent is zero, and interest (or profit) is impossible in a world of instantaneous production.

Competition assures

$$(12)\quad \mathbf{P}=[P_j]$$
$$=[Wa_{0j}+\sum_{i=1}^{n} P_ia_{ij}+0+0]>0$$
$$=W[A_{0j}]=\mathbf{WA}_0$$

where

$$(13)\quad \mathbf{A}_0=[A_{0j}]=\mathbf{a}_0[\mathbf{I}-\mathbf{a}]^{-1}>0$$
$$=\mathbf{a}_0+\mathbf{a}_0\mathbf{a}+\mathbf{a}_0\mathbf{a}^2+\ldots,$$

a convergent series.

Positivity and convergence in equation (13) is guaranteed by equation (11) plus the postulate that every good must indirectly, if not directly, require some labor if it is to be a good worth talking about in the rude state.

That the real wage can just buy the iron ration of subsistence was assured by Equations (11)–(13), which imply

$$(14)\quad \mathbf{Pm}=W=(\mathbf{A}_0\mathbf{m})W,\qquad \mathbf{A}_0\mathbf{m}=1$$

Incidentally, (14) tends to vindicate the empirical usefulness of Smith's notion of "labour command theory of value," as against Ricardo's semantic objections.

Stationarity of the rude states' population now follows from equation (7), which takes the form in the rude state of

$$(15)\quad (L_{t+1}-L_t)/L_t=f[1-\mathbf{A}_0\mathbf{m}]$$
$$=f[0]=0$$

Smith's identification of *net national product* in the rude state with wages only, or with the susbsistence consumptions of the workers, is verified:

$$(16)\qquad NNP=WL+0+0$$
$$=\sum_{1}^{n} P_jC_j=(\mathbf{A}_0\mathbf{C})W$$
$$=(\mathbf{A}_0\mathbf{m})WL$$

Investment and Malthusian Growth

Smith quickly turns the page on his rude state in which the labor theory of value holds. By the division of labor or otherwise, let some set of the elements of $(\mathbf{a}_0, \mathbf{a}, \mathbf{m})$ decrease. That raises equation (11)'s determinant from zero to positive. That raises the real wage above the subsistence level. That causes population initially to grow at an endogenous positive rate, like $(1+g)^t$. If we still keep production instantaneous, capital and positive profits cannot yet occur. The workers get all the fruits of the invention, and devote part of that fruit to procreation and longevity. Now

$$(17) \qquad \mathbf{Pm} < W,\ \mathbf{A_0m} < 1,\ \mathbf{C} \geq \mathbf{mL}$$

$$(L_{t+1} - L_t)/L_t = g = f[1 - \mathbf{A_0m}] > 0$$

$$L(t) = L(0)(1+g)^t,\ q(t) = q(0)(1+g)^t, \ldots, \quad t \geq 1$$

This initial state of exponential growth, à la Malthus (1798) and von Neumann (1932), must begin to decelerate once land becomes scarce. Eventually, workers elbow each other, trample down fields, and so forth. Land must be rationed by positive rentals, which for Smith were to go to the private appropriaters of land, selling their scarce inputs in a competitive market.

As L grows more and more relative to the fixed total of land, $\Sigma_1^n T_j = T$, positive rent income arises. Depending upon how landowners spend their rent incomes on consumption goods, and workers their surplus wages on goods, an equilibrium will emerge at each level of $(T, L, C_1, \ldots, C_n)$ for all prices $(P_1, \ldots, P_n, W, R)$. Smith's resolution of each P_j into W and R components was essentially correct, despite doubts in Marx (1885). And, even in the absence of profit and differences in time-phasing of production, Smith's solution does contradict the attempt in Ricardo (1817) to measure price ratios in terms of goods' labor contents alone.

Equilibrium Restored

At any stage of growth, for the given available technology and land, T, and for any prescribed pattern of feasible total consumption, $(C_1, \ldots, C_n)$, one can solve the planner's efficiency problem of minimizing needed total labor, L:

$$(18) \quad L = M(T; C_1, \ldots, C_n), \qquad C_i \geq 0$$

$$= \underset{T_i, V_i}{\mathrm{Min}} \sum_1^n a_{0j} V_j, \quad \text{subject to}$$

$$\sum_1^n a_{ij} V_j - F_i[T_i, V_i] + C_i \leq 0, \quad (i = 1, \ldots, n)$$

$$\sum_1^n T_j - T \leq 0,\ V_i \geq 0,\ T_i \geq 0$$

This is a standard problem in nonlinear programming, as in Kuhn and Tucker (1951). On the assumption that every good needs something of both land and variable factors, the necessary and sufficient conditions for the solution can be written down in terms of equalities involving "dual variables," or Lagrangean multipliers, or "shadow prices," which are interpretable as the non-negative price ratios $[P_1/W, \ldots, P_n/W, R/W]$, where R stands for the rental of land. (If T is a column vector of lands, R will be a row vector of rentals.) The unique conditions of equilibrium involve for scalar T,

$$(19) \quad (P_j/W)\,\partial F_j[T_j/V_j, 1]/\partial V_j = a_{0j} + \sum_1^n (P_i/W) a_{ij}$$

$$(P_j/W)\,\partial F_j[T_j/V_j, 1]/\partial T_j = R/W, \quad (j = 1, \ldots, n)$$

$$\sum_1^n a_{ij} V_j - F_i[T_i, V_i] = C_i, \qquad (i = 1, \ldots, n)$$

$$\sum_1^n T_j \leq T,\ R\left(T - \sum_1^n T_j\right) = 0,\ T_j > 0$$

These are $3n + 1$ independent equations that are just sufficient to determine the $3n + 1$ unknowns of the problem: $(V_1, \ldots, V_n; T_1, \ldots, T_n; P_1/W, \ldots, P_n/W, R/W)$. But equation (19), aside from having the planner's optimality interpretation, are precisely the *competitive* equilibrium conditions under Smith's postulated production conditions.

This identifies a valid element in Smith's INVISIBLE HAND doctrine: *self-interest, under perfect conditions of competition, can organize a society's production efficiently*. (But, there need be nothing ethically optimal about the $[C_i]$ specifications and their allocations among the rich and poor, the healthy and the halt!)

We indicate Smith's resolution of the price of every good into its total wage and rent components by deriving from (18) each good's total-land-and-labor requirements. We solve for the respective pairs:

$$(20)\ L_1^* = M(T_1^*; C_1, 0, \ldots, 0) \leftrightarrow C_1 = \phi_1[T_1^*, L_1^*]$$
$$L_2^* = M(T_2^*; 0, C_2, \ldots, 0) \leftrightarrow C_2 = \phi_2[T_2^*, L_2^*]$$
$$L_n^* = M(T_n^*; 0, 0, \ldots, C_n) \leftrightarrow C_n = \phi_n[T_n^*, L_n^*]$$

These $\phi_j[\ ]$ functions give the totals of land and labor required, directly *and* indirectly, to produce a net amount of each consumption good. These Smithian functions, never before written down explicitly in quite this way, are concave and first-degree-homogeneous; if the $F_j[\ ]$ functions are smoothly differentiable, as even Ricardo assumes in his arithmetic examples, so too will be the $\phi_j[\ ]$ functions. Hence, as in Shephard (1953), they will have dual unit-cost functions

$$(21)\ \phi_j^*[R, W] = \underset{T_j, V_j}{\text{Min}} \{(RT_j + WV_j)/\phi_j[T_j, V_j]\}$$

The ϕ_j^* functions have all the concavity, homogeneity, and differentiability properties of the $\phi_j[\ ]$ functions.

So, we sustain Smith against the objection that his eclectic breakdown of prices into wage and rent components is a trivial, surface relation. We write down for Smith:

$$(22)\quad P = \phi_j^*[R, W] + 0 = R\partial\phi_j^*[R, W]/\partial R + W\partial\phi_j^*[R, W]/\partial W$$

These partial equilibrium relations are well-determined by Smith's relations of general equilibrium in equation (19).

Finally, we solve for the new Smithian steady state of zero population growth after diminishing returns has brought the post-invention wage rate down to the subsistence level: we seek the L^* root of

$$(23)\quad L = M(T; m_1L + \gamma_1, \ldots, m_nL + \gamma_n)$$

where $(\gamma_1, \ldots, \gamma_n)$ represents landowners' choice of composition of their consumption goods. As Malthus realized, the equilibrium population will be larger or smaller depending upon whether rent collectors tend to spend their incomes on goods of high or low "labor intensity." Thus, their demand for "retainers" will mean greater L^* than will their demand for food or for hunting grounds.

In long-run equilibrium states where (13) holds and the real wage is at the subsistence level, the Physiocratic Land Theory of Value holds, as described in "A Modern Treatment of the Ricardian Economy" (Samuelson 1959). Landlords are faced by a linear budget constraint in choosing their γ's, namely:

$$(24)\qquad \tau_1\gamma_1 + \ldots + \tau_n\gamma_n = T$$

where the $[\tau_j]$ coefficients involve the total "socially necessary land" involved in each C_j's production, directly and indirectly and after including the land needed to produce the needed labor's subsistence.

Realistic Time-Phasing of Production

Since output is not instantaneously producible from inputs, inventories of raw materials and of subsistence wage goods are needed for steady-state production and for growth. Smith correctly recognized that the rate at which capitalist owners of such capital goods would be willing to save in order to "accumulate" them would set a limit on the system's growth and thereby generate a positive profit rate. With land fixed, new inventions ceasing, and population growing whenever the real wage exceeds subsistence, Smith correctly saw that continued saving and accumulation—contrived by capitalists' consuming less of their current profits than is available to them—must eventually induce a falling trend in the rate of profit. Finally, at a zero profit rate (over and above stochastic average losses) or at some low positive rate below which decumulation will occur, Smith's system reaches its longest-run equilibrium.

Let r^* be Smith's long-run, low positive rate of profit at which capitalists and landowners will spend all their incomes on current consumptions. With land fixed at T, no new inventions and no change in workers' subsistence (m_i), Smith correctly wrote his equilibrium in a tripartite breakdown of national income and each competitive price into wages, rents, and profits. His complete system becomes:

$$(25a)\quad F_i[T_i, V_i] - \sum_1^n a_{ij}V_j = m_i L + \gamma_i, \quad (i = 1, \ldots, n)$$

$$(25b)\quad P_j \partial F_j[T_j/V_j, 1]/\partial V_j = (Wa_{0j} + \sum_1^n P_i a_{ij})(1 + r^*)$$

$$(25c)\quad P_j \partial F_j[T_j/V_j, 1]/\partial T_j = R(1 + r^*), \quad (j = 1, \ldots, n)$$

$$(25d)\quad \sum_1^n a_{ij}V_j = L, \quad \sum_1^n T_j \leq T$$

$$(25e)\quad \sum_1^n P_j m_j = W > 0, \quad V_j \geq 0, \quad T_j \geq 0, \quad P_j \geq 0, \quad R \geq 0$$

For r^* and $\mathbf{m}$ sufficiently small, and for T and the ratios of nonworkers' taste parameters given $[\gamma_i/\gamma_1]$, these are $3n + 3$ equations for the equal number of unknowns: n V's, n T's, n (P/W)'s, γ_1, R/W, L. A meaningful solution is guaranteed to exist by virtue of the postulated properties for $F_j[\;]$.

Independently of the (γ_i) and (m_i) parameters, there is always a factor-price-frontier tradeoff between the real wage in terms of any good, W/P_j, the real rent, R/P_j, and the profit rate, r^*:

$$(26)\quad W/P_j = -\psi_j[R/P_j; r^*], \quad \partial\psi_j/\partial r^* > 0$$
$$\partial\psi_j/\partial(R/P_j) > 0, \quad \partial^2\psi_j/\partial(R/P_j)^2 \leq 0$$

For $r^* = 0$, $\psi_j[\;;0]$ is derivable from equating to unity $\phi_j^*[R/P_j, W/P_j)$. For $r^* > 0$ and all inputs used up in each single use, replacing the true $F_j[\;]$ functions by $(1 + r^*)^{-1}F_j[\;]$ will give rise to new $\phi_j[\;]$ and $\phi_j^*[\;]$ functions exactly as in equations (18) to (22). Then the fundamental factor-price frontiers defined by Smith's system can be defined by

$$(27)\quad \phi_j^*[R/P_j, W/P_j; 1 + r^*] = 1$$

For fixed $1 + r^*$, (27) defines convex contours.

With $r^* > 0$, equation (24)'s τ's have to be marked up, but are still constants so long as the m's are constants.

Prior to the system's having settled down into its long-run, time-phased steady state, one can provide for Smith's model an *endogenous* process of growth. Recognize the nonsimultaneous character of (1), and the need for capital inventories implied by such time phasing. So long as the initial rupture from the rude state is so recent that land is still redundant and rent zero, the system can grow in an initial golden age. Its rate of balanced exponential growth and the accompanying intermediate-run rate of interest or profit will provide the endogenous roots at which the supply of saving out of capitalists' profits are just large enough to provide the inventories for widening of capital goods and the advancing of wage goods for the multiplying population. If (7)'s population-growth function f[ ], is given; if (6)'s $[m_i]$ and (23)'s (γ_i) for nonlaborers are known; and finally if the fraction of profit that will be saved is a known function of the interest rate $s[r]$—then there will be an intermediate growth and profit rate, $(g^\dagger, r^\dagger)$, at which golden-age saving will equal golden-age warranted investment. Had Smith been able to write down the full conditions of this transient golden-age equilibrium, he would have anticipated Marx's expanded-reproduction tableaux of *Capital, Vol. II* and would have provided Harrod and Domar with an endogenous natural rate of growth.

Needless to say, once exponential growth runs into the constraint of scarce good land, positive rent will have to be reckoned with and recourse to ever-worse land, or ever-more-crowded best land, will imply a steadily dropping growth rate and a steady fall in the profit or the wage rate (or, most probably in both), as the post-rude *cheerful* state sinks into Smith's long-run *dull* state.